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Introduction to Linear Time-Invariant Systems

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Signal spaces

Discrete-time signals (sequences)

$$u : n \in \mathbb{Z} \to u_n \in \mathbb{C}$$

– If $\forall n \in \mathbb{N}, u_n \in \mathbb{R}$, then u is referred to as a *real* sequence

Continuous-time signals (functions)

$$f : x \in \mathbb{R} \to f(x) \in \mathbb{C}$$

– If $\forall x \in \mathbb{R}, f(x) \in \mathbb{R}$, then f is referred to as a *real* function

Signals with bounded support

Finite signals (FS): for a given $N \in \mathbb{N}$,

$$u: n \in \{0,1,\dots,N-1\} \to u_n \in \mathbb{C}$$

Equivalently, $u \in \mathbb{C}^N$

If $u \in \mathbb{R}^N$, we have a *real* FS

Periodic signals (PS):

$$f: x \in \left[-\frac{1}{2},\frac{1}{2}\right[\to f(x) \in \mathbb{C}$$

As for functions, if $\forall x \left[-\frac{1}{2},\frac{1}{2}\right[, f(x) \in \mathbb{R}$, we say that f is a *real* PS

Sequences spaces

Bounded sequences

$$u \in l^{\infty} \Leftrightarrow \|u\|_{\infty} = \sup_{n}\{|u_n|\} < +\infty$$

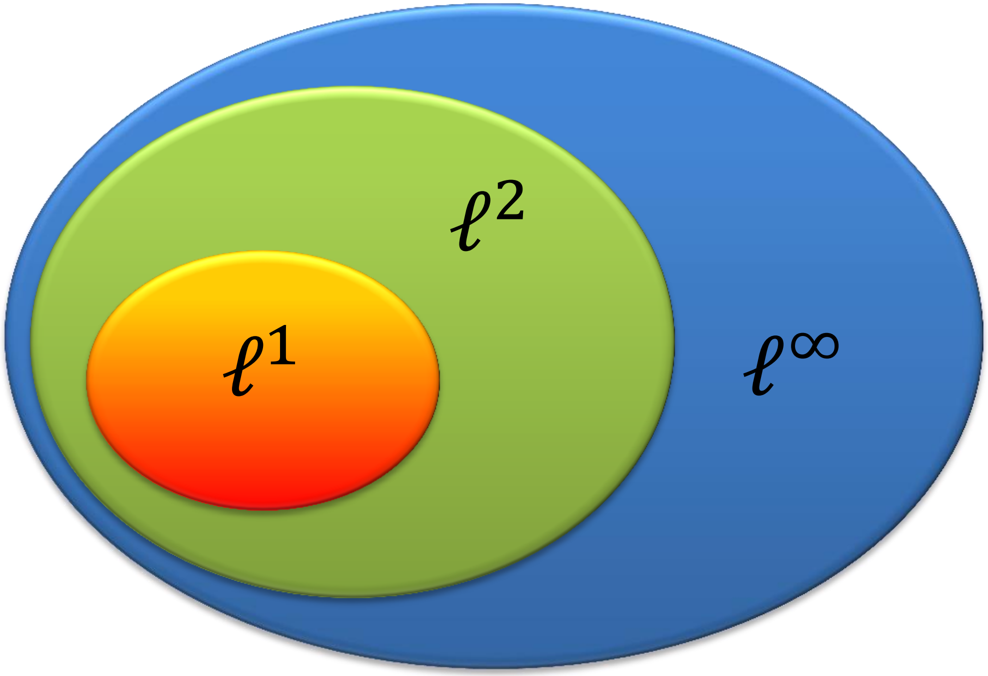
Square-summable sequences

$$u \in l^{2} \Leftrightarrow \|u\|_{2} = \sqrt{\sum_n |u_n|^2} < +\infty$$

Absolutely summable sequences

$$u \in l^{1} \Leftrightarrow \|u\|_{1} = \sum_{n} |u_n| < +\infty$$

Sequences spaces



$$\ell^1 \subset \ell^2 \subset \ell^\infty$$

$$u \in \ell^2, v \in \ell^2 \Rightarrow u \cdot v \in \ell^1$$
$$u \in \ell^1, v \in \ell^\infty \Rightarrow u \cdot v \in \ell^\infty$$

Convolution

- We have the following table

$*$	ℓ^1	ℓ^2	ℓ^∞
ℓ^1	ℓ^1	ℓ^2	ℓ^∞
ℓ^2	ℓ^2	ℓ^∞	$-$
ℓ^∞	ℓ^∞	$-$	$-$

- Practical rule:

$$\ell^1 * \ell^p \to \ell^p$$
$$\ell^2 * \ell^2 \to \ell^\infty$$

- The other cases, the convergence is not guaranteed (example : convolution of 2 constant-valued series)

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Signal spaces

$L^1(\mathbb{R})$: **absolutely integrable functions**

$$f \in L^1(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} |f(x)|\mathrm{d}x < +\infty$$

$L^2(\mathbb{R})$: **finite energy (square-integrable) functions**

$$f \in L^2(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} |f(x)|^2\mathrm{d}x < +\infty$$

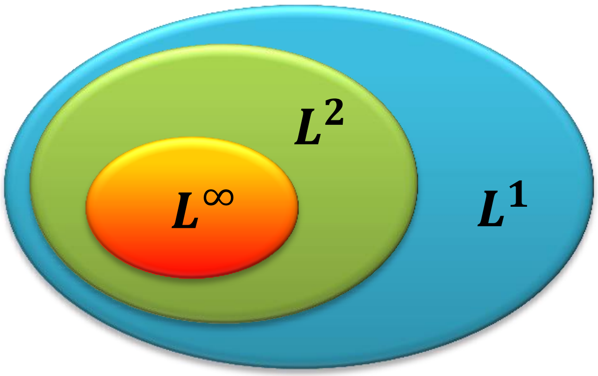
$L^\infty(\mathbb{R})$: **bounded functions**

$f \in L^\infty(\mathbb{R}) \Leftrightarrow \exists C \in \mathbb{R}: |f(x)| \leq C$ *a.e.*

Signal spaces for finite signals

- Any finite signal belongs to a normed space, namely, $\mathbb{C}^N$ or $\mathbb{R}^N$
- For periodic signals, we have

$$L^\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right) \subset L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right) \subset L^1\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right)$$



Systems

- A system is an operator transforming a signal into another signal
- Typically, we consider systems that transform functions into functions, sequences into sequences, PS into PS and FS into FS (exceptions: A/D and D/A converters)
- Continuous system:
  $T: f \rightarrow g = T[f]$
- Discrete system:
  $T: u \rightarrow v = T[u]$
- Formally, the same notation

Linear Time-Invariant (LTI) Systems

1. Linearity:

$$\forall \alpha, f: \ T[\alpha f] = \alpha T[f]$$
$$\forall f_1, f_2: \ T[f_1 + f_2] = T[f_1] + T[f_2]$$

2. Time-invariance:

$$\forall f, \Delta, \qquad T[f] = g \Rightarrow T\left[f^{\Delta}\right] = g^{\Delta}$$

Where **the notation $\boldsymbol{f^{\Delta}}$ stands for a shifted version of $\boldsymbol{f}$**: $f^{\Delta}(t) = f(t - \Delta)$

The possible values for Δ and the meaning of $(t - \Delta)$ depend on the signal space.

Examples

$$T[u] = v$$

$$v_n = u_n + u_{n-1} + 3u_{n+1}$$

$$v_n = u_{2n}$$

$$v_n = \max\{u_n, u_{n-1}, u_{n+1}\}$$

$$T[f] = g$$

$$g(t) = \int_{t-1/2}^{t+1/2} f(x)dx$$

$$g(t) = f(t-1)$$

$$g(t) = f(t)\cos(2\pi f_0 t + \phi)$$

Impulse response of a LTI

$$\delta_n = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$$

Impulse response of T : $h = T[\delta]$

Hypothesis : $h \in \ell^1$

Linear system as convolution: use $u = \sum_n u_n \delta_n$

$$v = T[u] = T\left[\sum_n u_n \delta^n\right] = \sum_n u_n T[\delta^n] = \sum_n u_n h^n$$

$$v_m = \sum_n u_n h_m^n = \sum_n u_n h_{m-n} = (u * h)_m$$

Stable LTI

- We consider the BIBO-stable systems
  - BIBO: bounded input-bounded output

- Necessary and sufficient condition:

$$h \in \ell^1$$
$$h \in L^1$$

Fourier waves

- Fourier waves (FW) at frequency ν have the following expression :

In $\mathbb{Z}$, $n \to e^{2i\pi\nu n}$ with $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[$

In $\mathbb{R}$, $x \to e^{2i\pi\nu x}$ with $\nu \in \mathbb{R}$

An LTI always transforms a FW into a FW with the same frequency

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Proof (sequences)

$$u_n = e^{2i\pi n\nu}$$
$$u_n^m = e^{2i\pi(n-m)\nu}$$

$$(T[u])_n = \sum_m h_m u_n^m = \sum_m h_m e^{2i\pi(n-m)\nu} =$$
$$= e^{2i\pi n\nu} \sum_m h_m e^{-2i\pi m\nu} = u_n \, \hat{h}(\nu)$$

$$\hat{h}(\nu) = \sum_m h_m e^{-2i\pi m\nu}$$

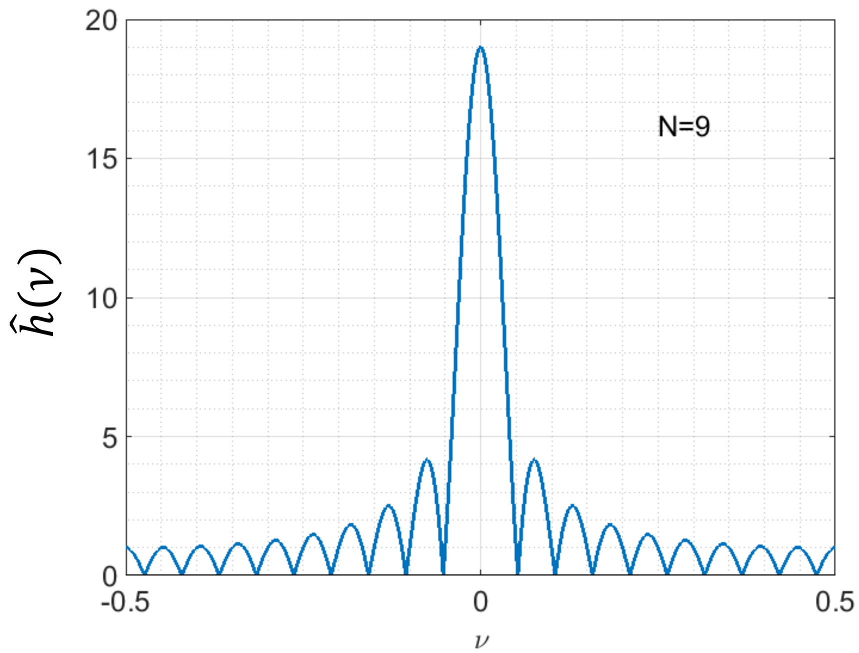
Example

$$h_n = \begin{cases} 1 \text{ if } n \in \{-N, -N+1, \dots, N-1, N\} \\ 0 \text{ otherwise} \end{cases}$$

Find $\hat{h}(\nu)$

$$\hat{h}(\nu) = \sum_{m=-N}^{N} e^{-2i\pi\nu m} =$$

$$= \begin{cases} 2N+1 & \text{if } \nu = 0 \\ \frac{\sin[(2N+1)\pi\nu]}{\sin \pi\nu} & \text{otherwise} \end{cases}$$



Continuous systems

Using the Dirac's delta, we can extend the previous results to continuous systems:

$$h = T[\delta]$$

$$g(x) = T[f](x) = \int h(t)f(x-t)dt$$

$$T\left[e^{2i\pi t\nu}\right] = \hat{h}(\nu)\, e^{2i\pi t\nu}$$

$$\hat{h}(\nu) = \int_{\mathbb{R}} h(t)e^{-2i\pi\nu t}dt$$

Summary

- If T is an LTI system:
  - It is characterized by its impulse response.
  - It is characterized by its frequency response.
- The impulse response defines the operation of the LTI via the convolution
- The frequency response defines the LTI via its operation on FWs

LTI on Finite signals

- To transpose the theory of LTI to these signals, it is enough to define :
  - Linear operations (obvious)
  - Time shift:

    Shift on $\{0, \dots, N-1\}$
  - Fourier waves

Time shift for a finite signal

$(u^m)_n = u_{n-m}$ for time sequences

For finite signals, $n - m$ can be outside $\{0,1, \dots\ N-1\}$

Solution : redefine the sum and difference in

$$\{0,1, \dots\ N-1\}$$

$$n \ \oplus_N m \ = \ (n+m) \text{ Mod } N$$

We continue to denote $n + m$ when there is no ambiguity

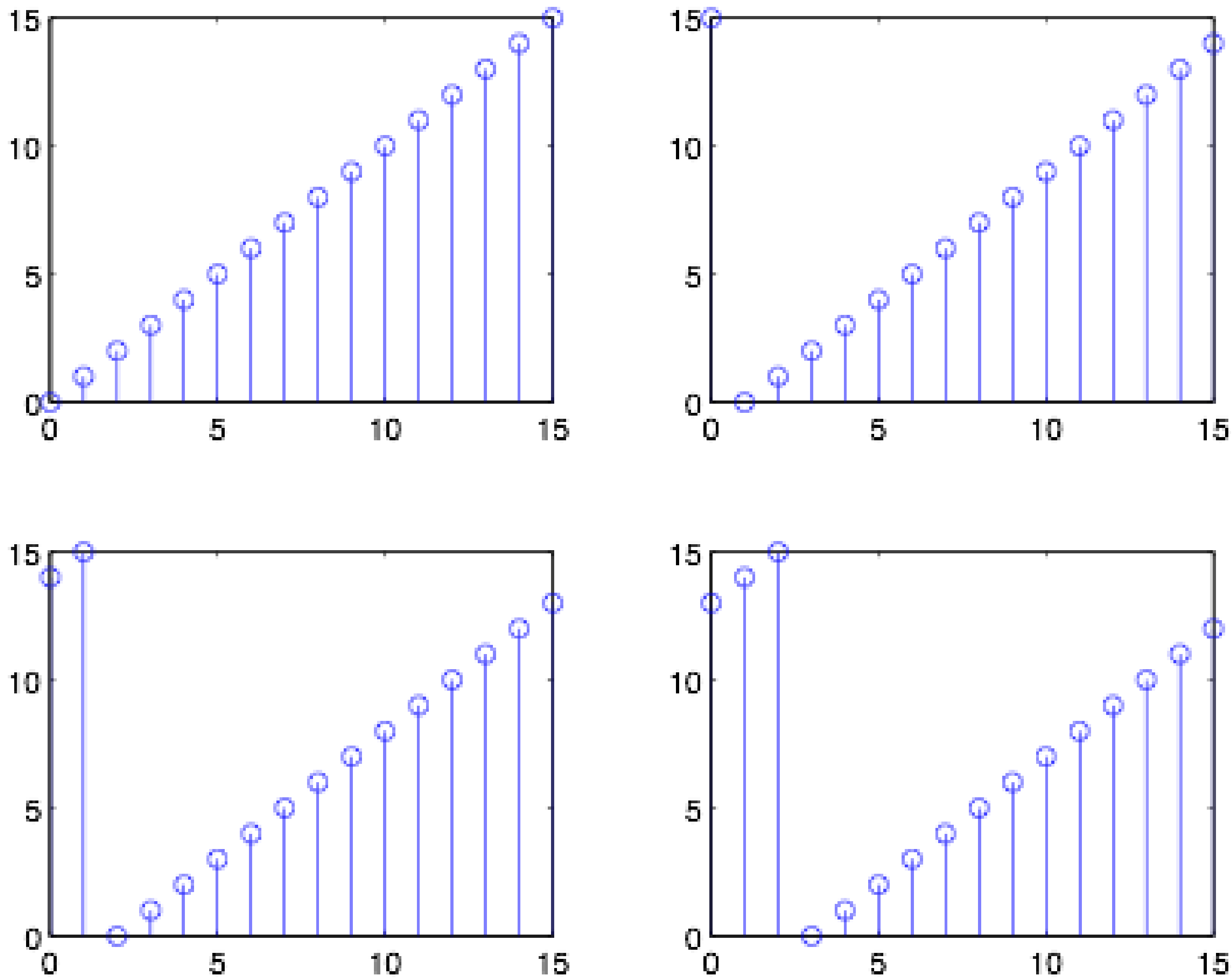
Time shift for a finite signal

$$(u^m)_n = u_{(n-m)\text{Mod N}}$$

Interpretation :

- Define the time sequence $\tilde{u}_n = u_{n \text{ Mod } N}$
- A period of $\tilde{u}$ coincides with u
- We define u^m as a period of $\tilde{u}^m$
- This guarantees $\forall m \in \mathbb{Z}, (u^m)^{-m} = u$
- The shift defined by this formula is called *Circular shift (circular permutation)*

Example of circular shift



LTI for finite signals

- For any finite signal u one can write :

$$u = u_0\delta^0 + u_1\delta^1 + \ldots u_{N-1}\delta^{N-1} = \Sigma\, u_n\delta^n$$

Then, by definition of LTI,

$$v = T[u] = \sum_{m=0}^{N-1} u_m T[\delta^m] = \sum_{m=0}^{N-1} u_m h^m$$

Where h^m is the time-shifted version of the impulse response $T[\delta]$

LTI for finite signals

- We find:

$$v_n = \sum_{m=0}^{N-1} u_m (h^m)_n = \sum_{m=0}^{N-1} u_m h_{(n-m)\ \mathrm{Mod}\ N} = (u \circledast_N h)_n$$

This is the *circular convolution* or convolution of finite signals

It has all the properties of an ordinary convolution :

Commutativity, associativity, linearity

Circular and ordinary convolution

Let u et h be finite signals of length *N*

Let $\tilde{u}$ be the sequence defined by periodization of u

$$\forall n \in \mathbb{Z}, \qquad \tilde{u}_n = u_{n \text{ Mod } N}$$

Let $v = u \circledast_N v$ be the circular convolution of u and h

$$v_n = \sum_{m=0}^{N-1} h_m u_{(n-m) \text{ Mod } N} = \sum_{m=0}^{N-1} h_m \tilde{u}_{(n-m)} = (h * \tilde{u})_n$$

Here we use the same notation h for a finite support sequence and a finite signal

The circular convolution is therefore the same as an ordinary convolution with a periodic (periodized) signal

Formulas for Fourier waves

- It is easy to check that the following functions satisfy the properties of the Fourier waves:

$\phi: n \in \{0,1,\ldots,N-1\} \rightarrow e^{2i\pi\frac{k}{N}n}$
with $k \in \{0,1,\ldots,N-1\}$

We call wave *frequency* the rational number $\frac{k}{N}$

Application of an LTI on Fourier waves

$$\phi_n = e^{2i\pi\frac{k}{N}n}$$

$$T[\phi]_n = \sum_{m=0}^{N-1} h_m \exp\{2i\,\pi\frac{k}{N}[(n-m)\mathrm{Mod}\,N]\}$$

$$(n-m)\ \mathrm{Mod}\ N = n - m + pN$$

$$T[\phi]_n = \sum_{m=0}^{N-1} h_m \exp\{2i\,\pi\frac{k}{N}[(n-m)]\} =$$

$$= \phi_n \sum_{m=0}^{N-1} h_m \exp(-2i\,\pi\frac{k}{N}m) = C(k)\phi_n$$