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Fourier Transform on $\mathbb{Z}$ (Discrete-Time Fourier Transform)

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Fourier waves reminders

- We have seen that any LTI acts on Fourier waves by multiplying them with a constant depending on the frequency. This constant is called frequency response.

- For a LTI on Z we have the formula:
- $C(\nu) = \sum_{m \in \mathbb{Z}} h_m e^{-2i\pi\nu n}$

C is the **frequency response** and h is the **impulse response**

Definition of the DTFT for a summable sequence

- If u is a sequence in ℓ^1, then we define its DTFT, which is a function on $[-1/2, 1/2[$, as:

$$\forall \nu \in [-\frac{1}{2},\frac{1}{2}[, \qquad \mathcal{F}[u](\nu) = \hat{u}(\nu) = \sum_{n\in\mathbb{Z}} u_n e^{-2i\pi\nu n}$$

- $\hat{u}(\nu)$ is a continuous function on $[-\frac{1}{2},\frac{1}{2}[$
- The formula can be immediately extended to $\nu \in \mathbb{R}$, where $\hat{u}$ is **periodic** and **continuous**

Some properties

$$u, v \in l^1; \nu_0 \in [-\frac{1}{2}, \frac{1}{2}[;$$

$$\phi: n \in \mathbb{Z} \to e^{2i\pi\nu_0 n}, \qquad \psi: \nu \in [-\frac{1}{2}, \frac{1}{2}[\to e^{-2i\pi m\nu}$$

$$u = \delta^m \Rightarrow \hat{u}(\nu) = \psi(\nu)$$

$$\mathcal{F}(u * v) = \hat{u}\hat{v}$$

$$\mathcal{F}(uv) = \hat{u} * \hat{v}$$

$$[\mathcal{F}(\phi.u)](\nu) = \hat{u}(\nu - \nu_0)$$

$$\mathcal{F}(u^m) = \hat{u}.\psi$$

$$u \text{ real} \Rightarrow \hat{u}(-\nu) = \overline{\hat{u}(\nu)}$$

$$u_n = u_{-n} \Rightarrow \hat{u}(-\nu) = \hat{u}(\nu)$$

u real and even $(u_n = u_{-n}) \Rightarrow \hat{u}$ real and even $\left(\hat{u}(-\nu) = \hat{u}(\nu)\right)$

(Hints for) proofs

$$\left(\delta^{k} * \delta^{p}\right)_n = \sum_{m\in\mathbb{Z}} \delta^{k}_{m}\delta^{p}_{n-m} = \delta_{n-k-p} = \left(\delta^{k+p}\right)_n$$

$$\mathcal{F}\left(\delta^{k} * \delta^{p}\right) = \mathcal{F}\left(\delta^{k+p}\right) = \mathcal{F}\left(\delta^{k}\right)\mathcal{F}(\delta^{p})$$

If $\forall \nu \in \left[-\frac{1}{2},\frac{1}{2}\right[,\ e_m(\nu) = e^{2i\pi\nu m}$, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e_m(\nu)\overline{e_n(\nu)}d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2i\pi\nu(m-n)}d\nu \ =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos[2\pi\nu(m-n)]\,d\nu + i\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin[2\pi\nu(m-n)]\,d\nu$$

$$= \delta_{m-n}$$

(Hints for) proofs

$$\mathcal{F}(\phi \cdot u)(\nu) = \sum_n u_n e^{2i\pi\nu_0 n} e^{-2i\pi\nu n} = \hat{u}(\nu - \nu_0)$$

$$\mathcal{F}(u^m) = \mathcal{F}(u * \delta^m) = \hat{u}\psi$$

$$u \text{ real} \Rightarrow \overline{\hat{u}(\nu)} = \overline{\sum_{m\in\mathbb{Z}} u_m e^{-2i\pi n\nu m}} = \sum_{m\in\mathbb{Z}} u_m e^{2i\pi n\nu m} = \hat{u}(-\nu)$$

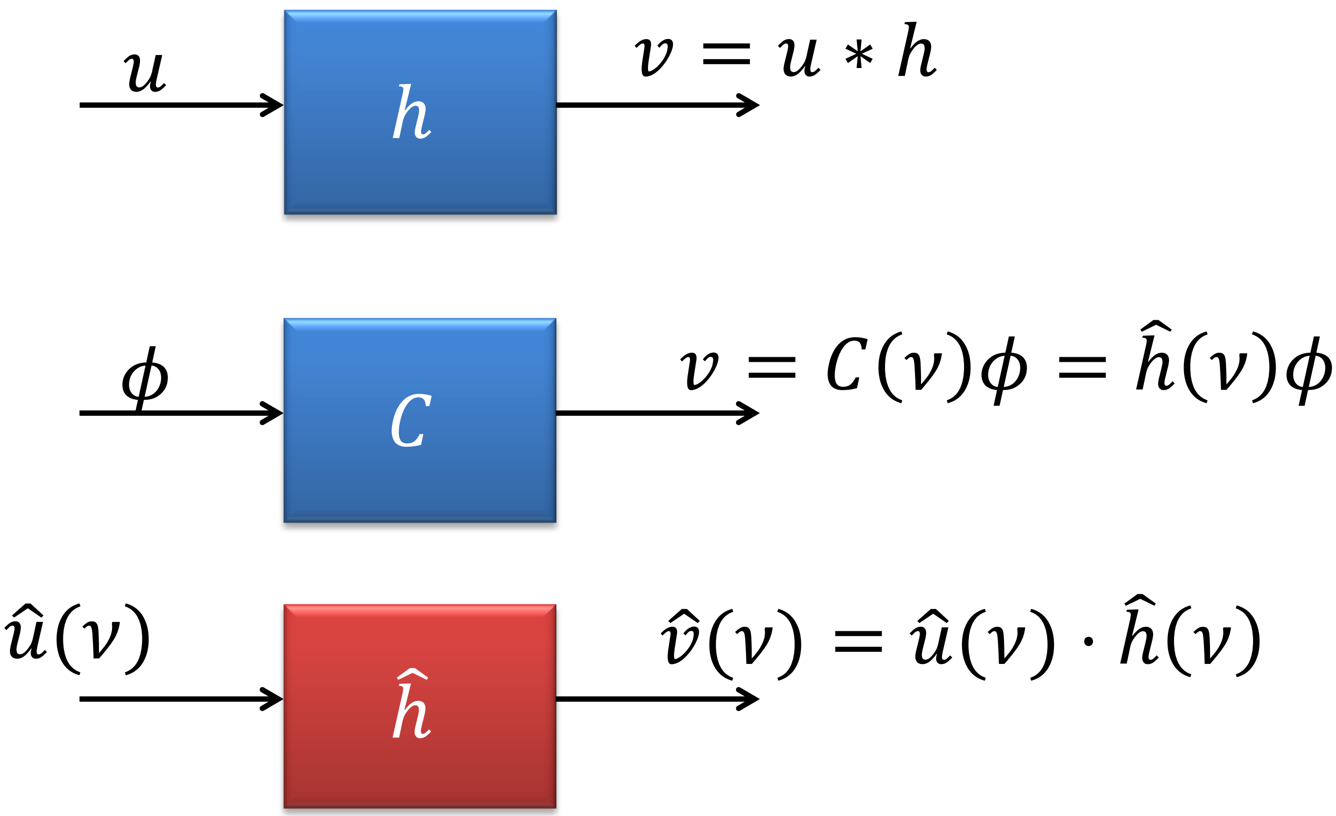
$$u_{-n} = u_n \Rightarrow \hat{u}(-\nu) = \sum_{m\in\mathbb{Z}} u_m e^{2i\pi n\nu m} = \sum_{n\in\mathbb{Z}} u_n e^{-2i\pi n\nu n}$$

$$= \hat{u}(\nu)$$

Interpretation in terms of LTI

- If $h \in l^1$, then the associated LTI T transforms a bounded sequence into a bounded sequence
- If $u \in l^1$ and $u = T[u]$, then $v = h * v \in l^1$
- $C(\nu) = \sum_{m\in\mathbb{Z}} h_m e^{-2i\pi\nu m} = \hat{h}(\nu)$
- $\hat{v} = \hat{u}\hat{h}$

Interpretation in terms of LTI



Extension to ℓ^2 and Parseval identity

- Extend the discrete time Fourier transform definition from ℓ^1 to ℓ^2
- Same notation: $\mathcal{F}[u] = \hat{u}(\nu)$
- Consider $L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right)$, the space of finite-energy periodical functions :

$$f \in L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right) \Leftrightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)|^2 dt < +\infty$$

Extension to ℓ^2 and Parseval identity

Theorem

- Let $\mathcal{F}$ be a linear application
  $\mathcal{F}: \ell^2 \to L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right[\right)$
  such that the restriction of $\mathcal{F}$ on ℓ^1 is the DTFT
- then :
  - $\mathcal{F}$ exists and is unique
  - $\mathcal{F}$ is a bijection between ℓ^2 and $L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right[\right)$
  - $\mathcal{F}$ is an isometry (Parseval identity) :

$$\sum_{n\in\mathbb{Z}} |u_n|^2 = \int_{-1/2}^{1/2} |\hat{u}(\nu)|^2 d\nu$$

Extension to ℓ^2 and Parseval identity

$\forall u \in \ell^2, \forall N \in \mathbb{N}$ we consider the following function sequence:

$$\hat{u}_N(\nu) = \sum_{n=-N}^{N} u_n e^{-2i\,\pi\nu n}$$

It can be shown that it converges to an element of $L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$

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Extension to ℓ^2 and Parseval identity

Proof only for finite support series

Let $e_m(\nu) = \exp(2i\pi\nu m)$

We have seen that: $\int_{-1/2}^{1/2} e_m\, \bar{e}_n d\nu = \delta_{n-m}$

$$\|\hat{u}\|_2^2 = \int_{-1/2}^{1/2} \hat{u}(\nu)\overline{\hat{u}(\nu)} d\nu =$$

$$\int_{-1/2}^{1/2} \Sigma_m u_m e_{-m} \Sigma_n \overline{u_n} e_n d\nu = \Sigma_{n,m} u_m \bar{u}_n \delta_{n-m}$$

$$= \|u\|_2^2$$

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$$\mathcal{F}(u * v) = \hat{u}\hat{v}$$

If v is summable

$$\mathcal{F}(uv) = \hat{u} * \hat{v}$$

$$[\mathcal{F}(\phi.u)](\nu) = \hat{u}(\nu - \nu_0)$$

$$\mathcal{F}(u^m) = \hat{u}.\psi$$

$$u \text{ real} \Rightarrow \hat{u}(-\nu) = \overline{\hat{u}(\nu)}$$

$$u_n = u_{-n} \Rightarrow \hat{u}(-\nu) = \hat{u}(\nu)$$

$$u \text{ real and } u_n = u_{-n} \Rightarrow \hat{u} \text{ real and } \hat{u}(-\nu) = \hat{u}(\nu)$$

Inversion theorem

If $u \in \ell^2$ and $\hat{u} = \mathcal{F}(u)$,

Then

$$\forall n \in \mathbb{Z}, u_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{u}(\nu) e^{2i\pi\nu n} d\nu$$

Proof for sequences of finite support :

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{u}(\nu) e^{2i\pi\nu n} d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m \in \mathbb{Z}} u_m \, e^{-2i\pi\nu m} e^{2i\pi\nu n} d\nu$$

$$= \sum_{m \in \mathbb{Z}} u_m \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2i\pi\nu m} e^{2i\pi\nu n} d\nu = u_n$$

Decrease to infinity and regularity

Let $k \in \mathbb{N}$ and

$$\sum_{n \in \mathbb{Z}} |n|^k |u_n| < +\infty$$

then $\hat{u} = \mathcal{F}[u]$ exists and $\hat{u} \in \mathcal{C}^k\left(\left[-\frac{1}{2}, \frac{1}{2}\right[\right)$, i.e.

$\hat{u}$ is k times continuously derivable.

If we denote by v_n^k the sequence of general term :

$$v_n^k = (-2i\pi n)^k u_n$$

then $v \in \ell^1$ and

$$\mathcal{F}[v^k](\nu) = \frac{\partial^k \hat{u}(\nu)}{\partial \nu^k}$$