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Z Transform

Notations

- IR : impulse response
- Causal sequence: h_n **null** for any $n < 0$
- Causal LTI if its IR is causal
- Anti-causal sequence: h_n **null** for any $n \geq 0$
- Anti-causal LTI if its IR is anti-causal
- Bilateral: neither causal nor anti-causal
- FIR: finite IR (finite support)
- IIR: infinite support for its IR

Causality properties

- A causal LTI does depend only on the past of its input signal.
- The convolution of two causal sequences is causal.
- Thus, the convolution of two causal LTI is a causal LTI

$(u * v)_n = \sum_{m \in \mathbb{Z}} u_m v_{n-m} = \sum_{m \geq 0} u_m v_{n-m}$, null if $n < 0$

- Any physical (real) system is causal
  - But if n is not a temporal variable or if the processing does not need to be real time, non-causal systems are realistic!

Z Transform (ZT)

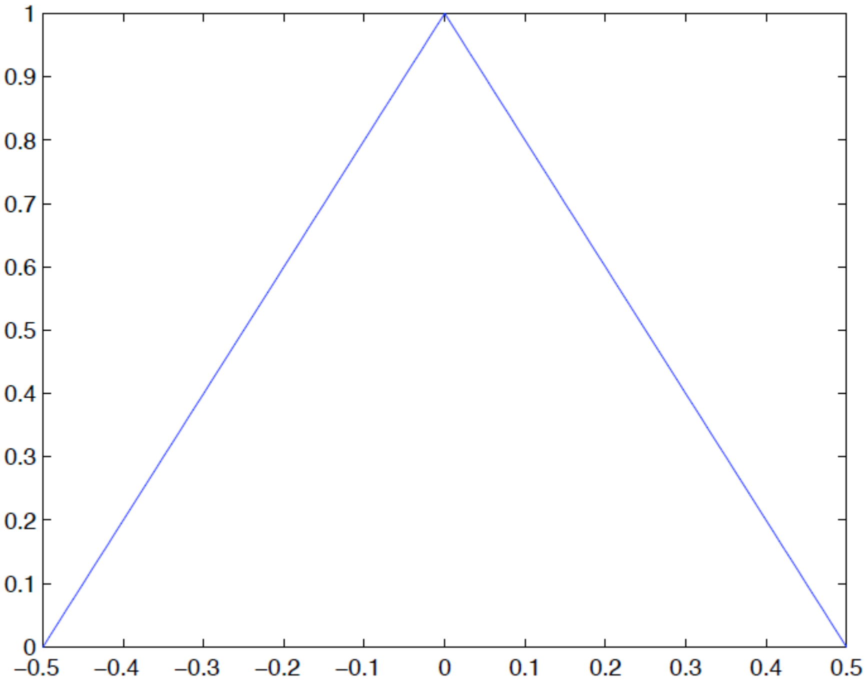
If $h \in \ell^1$, we can define its ZT (denoted by $H(z)$) as:

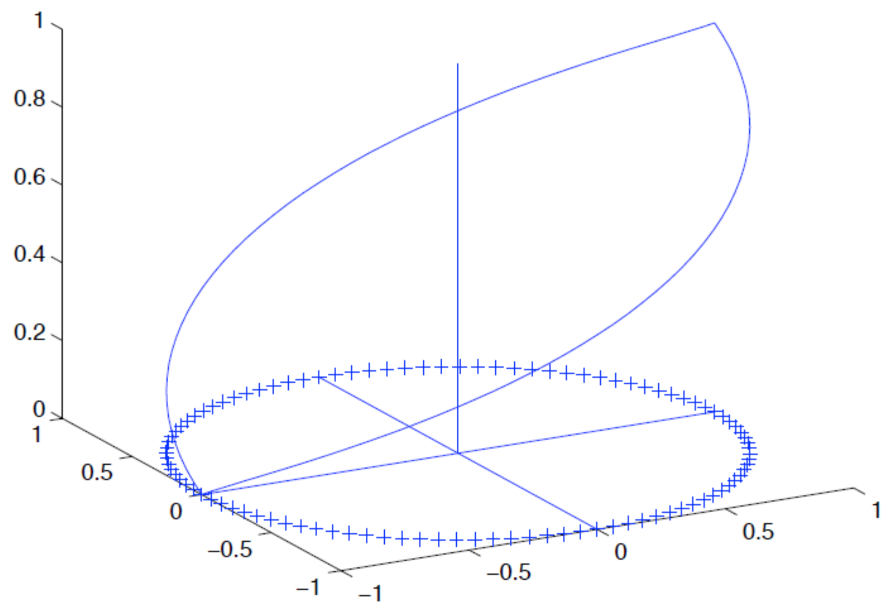
$$\forall z \in \mathbb{U}, \qquad \mathcal{Z}[h](z) = H(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n}$$

where $\mathbb{U} = \{z \in \mathbb{C}: |z| = 1\}$ (unit circle)

We observe that $\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[, \hat{h}(\nu) = H(z)\big|_{z=e^{2i\pi\nu}}$

and also $\forall z \in \mathbb{U}, \;\; H(z) = \hat{h}\left(\frac{\ln z}{2i\pi}\right)$





DTFT and ZT of the same sequence.
We have a graphical illustration of the frequency periodicity

ZT Properties

Thanks to the relation between the ZT and the DTFT, one can easily find the properties of the ZT:

Convolution : If $u, h \in \ell^1$, we know that $v = u * h \in \ell^1$ and we have therefore:

$$V(z) = U(z)H(z)$$

Inversion : If $u \in \ell^1$ and $U(z)$ is its ZT,

$$\forall n \in \mathbb{Z}, \qquad u_n = \int_{-1/2}^{1/2} H\left(e^{2i\pi\nu}\right) e^{2i\pi\nu} \mathrm{d}\nu$$

Therefore, the ZT is injective

ZT Properties

$$\forall u, v \in \ell^1, \qquad \forall m \in \mathbb{Z}, \qquad \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

Time shift:

$\forall z \in \mathbb{U},$

$$\mathcal{Z}[u^m](z) = \sum_{n \in Z} u_{n-m} z^{-n} = \sum_{k \in Z} u_k z^{-k-m} = z^{-m} U(z)$$

Linearity :

$$\forall z \in \mathbb{U}, \qquad \mathcal{Z}[\lambda_1 u + \lambda_2 v](z) = \lambda_1 U(z) + \lambda_2 V(z)$$

Examples of ZT

- If $h_n \neq 0 \Leftrightarrow n \in \{0,1,\ldots,N\}$, then its ZT is a polyomial in z^{-1}:

$$H(z) = \sum_{n=0}^{N} h_n z^{-n} = P(z^{-1})$$

- $h_0 = 1, h_1 = -\frac{1}{2}, h_n = 0 \; \forall n \in \mathbb{Z} - \{0,1\}$

$$H(z) = 1 - \frac{1}{2} z^{-1}$$

- $h_n = \left(\frac{1}{2}\right)^n$ if $n \geq 0$, and $h_n = 0$ otherwise

Find $H(z)$

Examples of ZT

$$H(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

- Remark that the last two sequences are IR of LTI inverse of one of each other...

Stable recursive filters

- A digital filter is a LTI on $\mathbb{Z}$
- A LTI is stable if $h \in \ell^1$ (BIBO stability – *Bounded input, bounded output*)
- A stable filter (LTI) is said **recursive** if

1. The input/output relationship is:

$$\begin{aligned} \forall n \in \mathbb{Z} \quad & b_0 y_n + b_1 y_{n-1} + \ldots + b_q y_{n-q} \\ = \quad & a_0 x_n + a_1 x_{n-1} + \ldots + a_p x_{n-p} \end{aligned}$$

2. Polynomials $P(z) = \sum_i a_i z^i$ et $Q(z) = \sum_i b_i z^i$ are co-prime

Stables recursive filters

- Allow to implement a larger class than FIR filters *in a finite number of computations*
- **In an exact manner** if the input is causal
  - But any acquired signal is causal, at least by delaying it
- With an **arbitrary precision** otherwise

ZT of an IR of a stable recursive filter

- If H is the ZT of the IR of a stable recursive filter then (with the previous notations)

$$H(z) = \frac{P(z^{-1})}{Q(z^{-1})}$$

- This means that the polynomial Q does not cancel on the unit circle.

Exponential decrease of the decomposition of a rational function

If P and Q are 2 co-prime polynomials and Q does not cancel on $\mathbb{U}$, then there exists a sequence h
and real positive numbers R_1, R_2, C_1, C_2 s.th.:

$$
\begin{aligned}
&0 < R_1 < 1 < R_2 \\
&\forall n \geq 0, \qquad |h_n| < C_1 R_1^n, \text{et } |h_{-n}| < C_2 R_2^{-n} \\
&\forall z \in \mathbb{C}: R_1 < |z| < R_2, \frac{P(z^{-1})}{Q(z^{-1})} = \sum_{n \in \mathbb{Z}} h_n z^{-n}
\end{aligned}
$$

Moreover, $h \in l^1$ et $\forall z \in \mathbb{U}, H(z) = \frac{P(z^{-1})}{Q(z^{-1})}$

Two important decompositions

Let $H(z) = \frac{1}{1-\alpha z^{-1}}$.

If $\alpha \in \mathbb{C}: |\alpha| < 1$, then

$$\forall z \in \mathbb{U}, \frac{1}{1 - \alpha z^{-1}} = \sum_{n \geq 0} \alpha^n z^{-n}$$

If $\alpha \in \mathbb{C}: |\alpha| > 1$, then

$\forall z \in \mathbb{U}$,

$$\frac{1}{1 - \alpha z^{-1}} = \frac{-\alpha^{-1} z}{-\alpha^{-1} z + 1} = -\sum_{n > 0} \alpha^{-n} z^n = -\sum_{m < 0} \alpha^m z^{-m}$$

If α is inside [outside] of the unit circle, $H(z)$ is the ZT of an [anti-]causal time sequence

A recurrence equation has a single summable solution

- For a recurrence equation, if Q has no zeros on the unit circle, then for any summable input x, there exists a unique solution $y \in \ell^1$ such that $y = x * h$, where h is the only summable time sequence whose ZT is P/Q.
- Thus , a recurrence equation defines a stable and recursive LTI (under the condition on Q).

A recurrence equation has a single summable solution

Unicity. Let y^1 and y^2 be 2 solutions associated with x :

$$\sum b_m (y^1)^m = \sum a_k x^k = \sum b_m (y^2)^m$$
$$\sum b_m z^{-m} Y^1(z) = Q(z^{-1}) Y^1(z) =$$
$$\sum b_m z^{-m} Y^2(z) = Q(z^{-1}) Y^2(z) \Rightarrow Y^1(z) = Y^2(z)$$

The injectivity guarantees that $y^1 = y^2$

Form. The previous theorem guarantees that if we define $H(z) = \frac{P(z^{-1})}{Q(z^{-1})}$, then $\exists h \in l^1 : \mathcal{Z}[h](z) = H(Z)$

Let $y = h * x$. Since we know that $y \in l^1$, we can thus compute the ZT.

A recurrence equation has a single sommable solution

$$y = h * x \Rightarrow Y(z) = H(z)X(z) = \frac{P(z^{-1})}{Q(z^{-1})}X(z)$$

$$Q(z^{-1})Y(z) = P(z^{-1})X(z)$$

$$\sum_m b_m z^{-m} Y(z) = \sum_k a_k z^{-k} X(z) \xrightarrow[\text{TZ-I}]{}$$

$$\sum_m b_m y_{n-m} = \sum_k a_k x_{n-k} \Rightarrow$$

y is the (unique) solution

Simplification of a recurrence equation

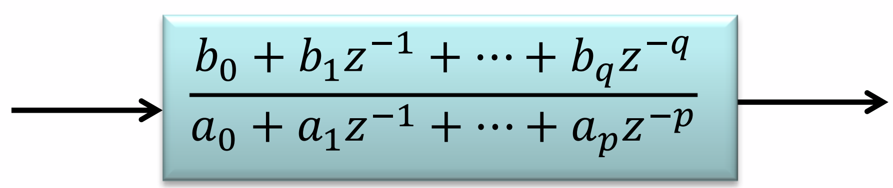
- Consider the (non trivial) recurrence equation

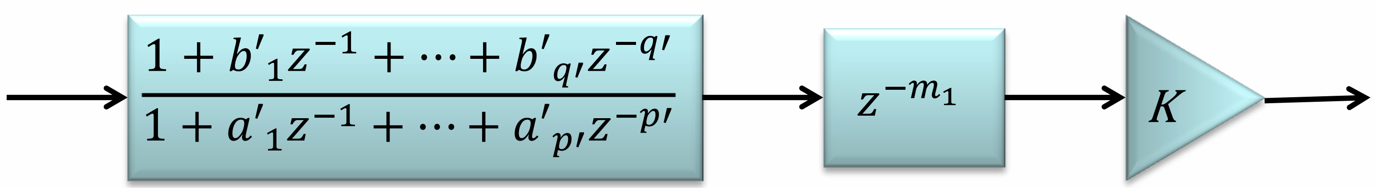
$$\forall n \in \mathbb{Z} \quad b_0 y_n + b_1 y_{n-1} + \ \dots + b_q y_{n-q} \quad = \quad a_0\, x_n + a_1 x_{n-1} + \ \dots + a_p x_{n-p}$$

- With simple translations of x and y, we can suppose that $b_0 = 1$ and a_0 different from zero.

- We can also replace a_i by a_i / a_0 and have the new $a_0 = 1$: this is equivalent to multiply y by a constant

- We shall make these simplifying hypotheses in the following.





Poles and zeros

- Let consider the followinf recursive stable LTI (a_0 and b_0 different from zero)

$$\forall n \in \mathbb{Z} \quad b_0 y_n + b_1 y_{n-1} + \ \dots + b_q y_{n-q} = a_0\, x_n + a_1 x_{n-1} + \ \dots + a_p x_{n-p}$$

- The zeros of the filter are the zeros of $P(z^{-1})$
- The poles of the filter are the zeros of $Q(z^{-1})$

They are also the zeros and the poles of $H(z)$

Poles and zeros : examples

$$y_n = x_n + \frac{1}{2}x_{n-1}$$
$$H(z) = 1 + \frac{1}{2}z^{-1}$$

Poles : none ;
zeros : $z_0 = -\frac{1}{2}$

$$y_n + \frac{1}{2}y_{n-1} = x_n$$
$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}$$

Poles : $\rho_0 = -\frac{1}{2}$;
zeros : none

$$y_n + \frac{1}{3}y_{n-1} = x_n - \frac{1}{4}x_{n-2}$$
$$H(z) = \frac{1 - \frac{1}{4}z^{-2}}{1 + \frac{1}{3}z^{-1}}$$

Poles : $\rho_0 = -\frac{1}{3}$;
zeros : $z_0 = \frac{1}{2},\ z_1 = -\frac{1}{2}$

Interpretation of the ZT module of a rational function

$$a_0 = 1, b_0 = 1, \qquad H(z) = \frac{P(z^{-1})}{Q(z^{-1})}$$

$$P(t) = \sum a_i t^i, \mathrm{Q}(t) = \sum b_i t^i$$

then $|H(z)| = \frac{\prod_i MA_i}{\prod_j MB_j}$

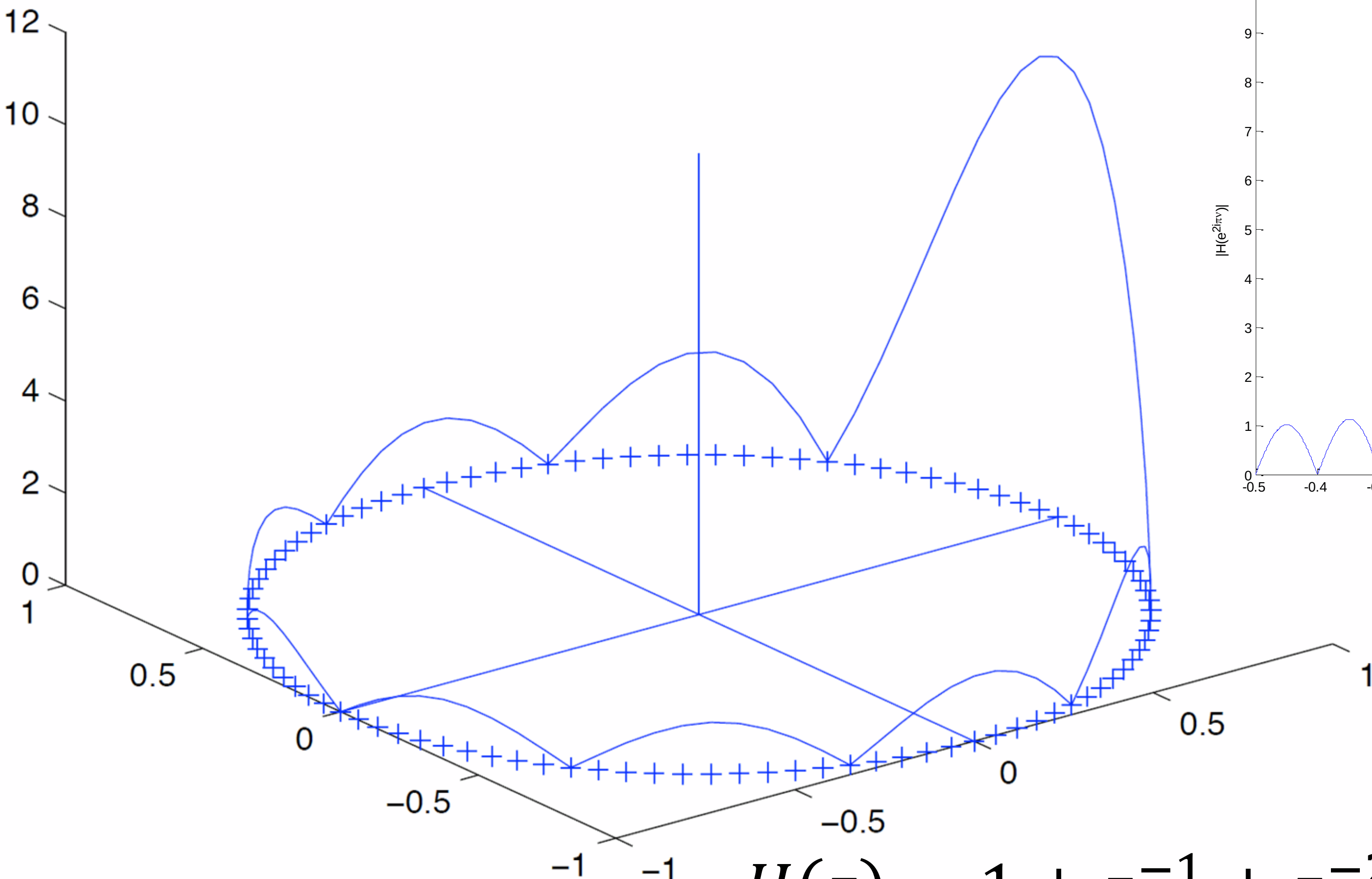
where M is the point of affix z, A_i, B_j are the points of affixes the zeros [poles] of the filter

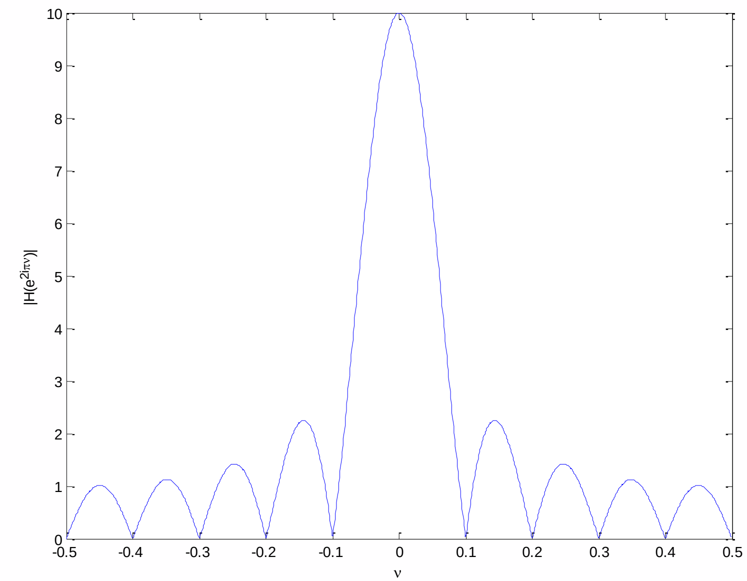
$$P(z) = a_p \Pi_i \left(z - \alpha_i^{-1}\right)$$

$$P(0) = a_0 = 1 = a_p \Pi_i \left(-\alpha_i^{-1}\right)$$

$$\left(\mathrm{z}^{-1} - \alpha_i^{-1}\right) = \frac{\alpha_i - z}{\alpha_i z} \quad \Rightarrow |\, P(z^{-1})| = \Pi_i |z - \alpha_i|$$

Example





$$H(z) = 1 + z^{-1} + z^{-2} + \ ... + z^{-N+1}$$

Amplitude of the ZT of the averaging filter (N=10 here). Its amplitude is zero as we cross an N-th root of the unity, except the root =1.

Inversion of a stable recursive filter

- We have a SRF with P and Q having no zero in $\mathbb{U}$:

$$\forall n \in \mathbb{Z} \quad b_0 y_n + b_1 y_{n-1} + \ \ldots + b_q y_{n-q}$$
$$= \quad a_0\, x_n + a_1 x_{n-1} + \ \ldots + a_p x_{n-p}$$

- Then the system is invertible and its inverse is a SRF with equation:

$$\forall n \in \mathbb{Z} \quad a_0 y_n + a_1 y_{n-1} + \ \ldots + a_q y_{n-q}$$
$$= \quad b_0\, x_n + b_1 x_{n-1} + \ \ldots + b_p x_{n-p}$$

- Remark: if $P(\cdot)$ had a zero in $\mathbb{U}$, the system would not be invertible

- Remark: the inverse of an FIR is an IIR

Causality

- An SRF ($b_0 = 1$) is causal iff all its poles (if any) have absolute value strictly less than one.

$$H(z) = \frac{P(z^{-1})}{Q(z^{-1})} = R(z^{-1}) + \sum_j \frac{\gamma_j}{1 - z^{-1}\beta_j}$$

H causal $\Leftrightarrow H - R = H_1$ causal

If some poles are outside the unity disk, for some $n < 0$,

$$h_n = \sum_{j \in B} \gamma_j \beta_j^n \quad (1)$$

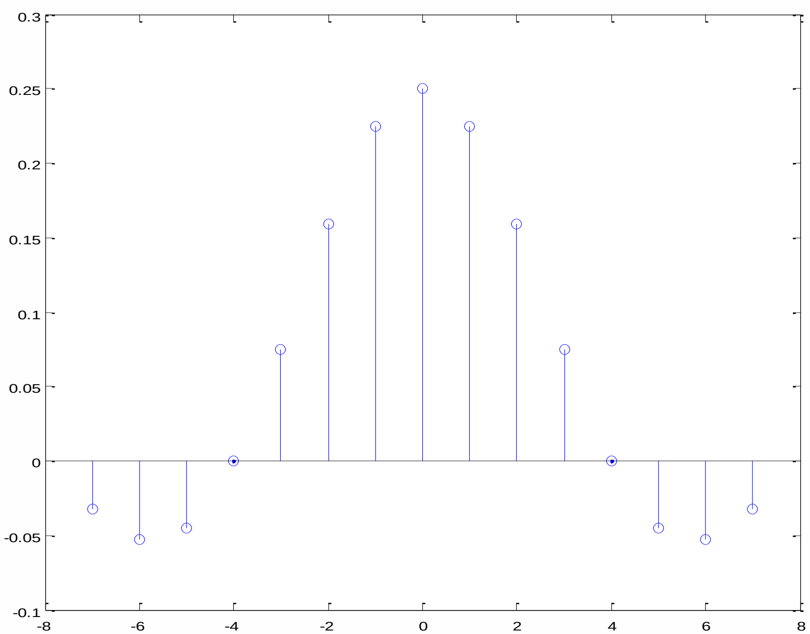
B set of poles outside $\mathbb{U}$

For $|n|$ large enough, in this sum there is a dominant term, so h cannot be causal

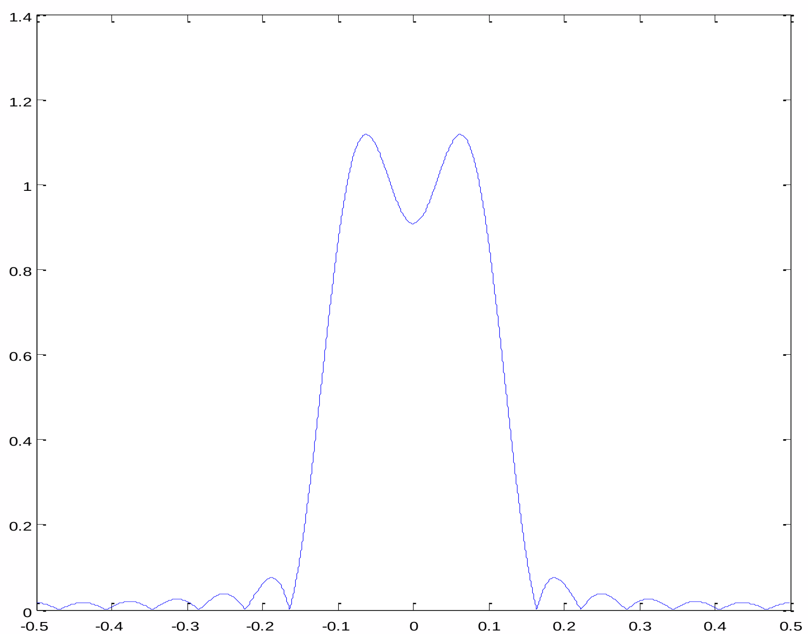
Filter synthesis: Window method

- h is a given IR. We want to approximate it through a FIR g satisfying:
$$g_n = 0 \text{ if } |n| > N$$
- The criterion to optimize is : $\int_{-1/2}^{1/2} \left|\hat{h}(\nu) - \hat{g}(\nu)\right|^2 \mathrm{d}\nu = \left\|\hat{h} - \hat{g}\right\|_2^2 = \|h - g\|_2^2 =$
$$\sum_{|n|\leq N} |h_n - g_n|^2 + \sum_{|n|>N} |h_n|^2$$
- The optimum is: $g_n = \begin{cases} 0 & \text{if } n > N \\ h_n & \text{if } n \leq N \end{cases}$, hence the name of the method
- To approximate an ideal low pass filter at frequency 1/8, $h_n = \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$
The square error decreases as $1/N$

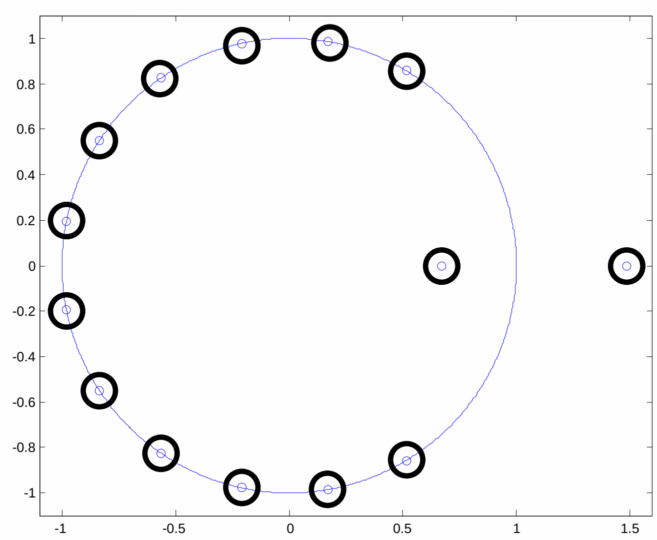
Filter synthesis: window method



g_n

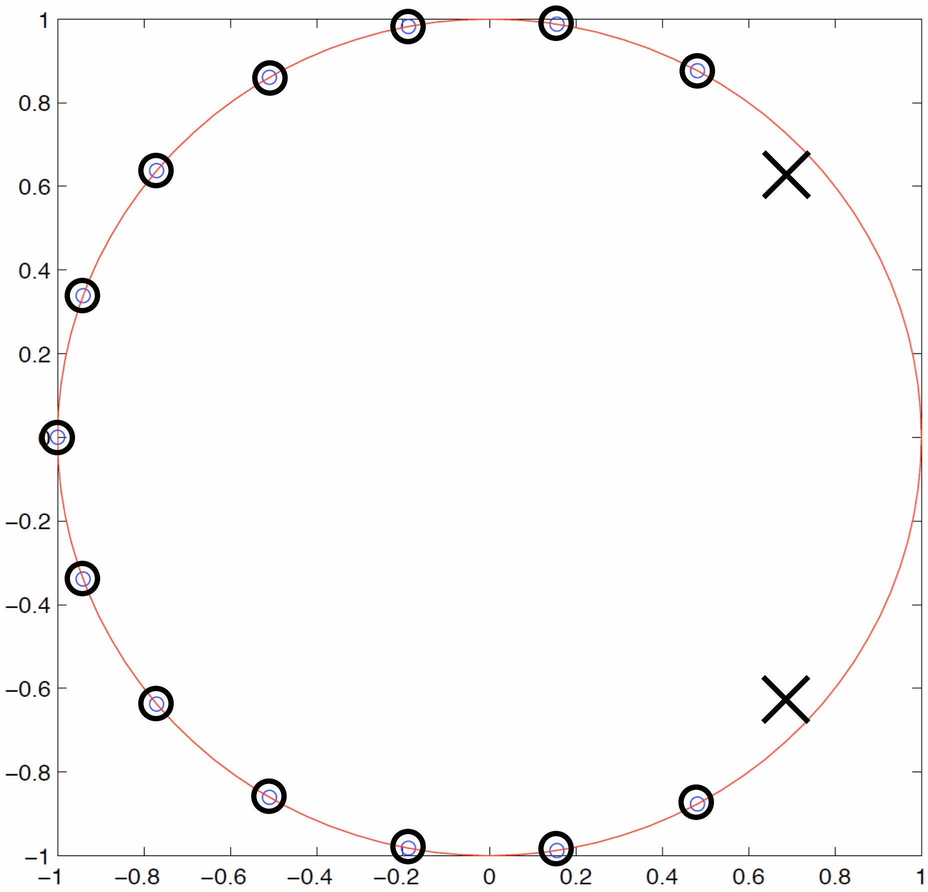


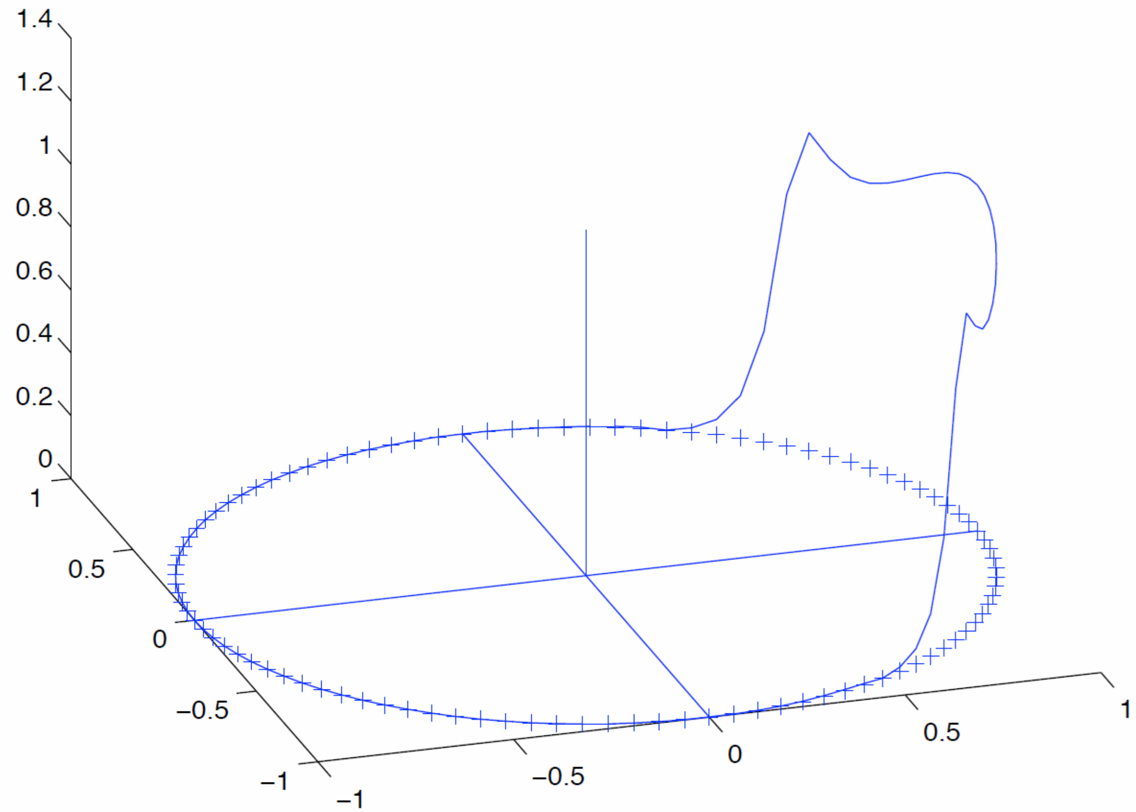
$|\hat{g}(\nu)|$



Zeros of $G(z)$

Filter synthesis by choice of the poles and zeros of a stable recursive LTI





With the same number of operations as in the window method, we get an approximation 3 times better ! (left: 2 poles ‘x’ and13 zeros ‘o’).