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Sampling

MN 901

Function spaces

$L^1(\mathbb{R})$: **absolutely integrable functions**

$$f \in L^1(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} |f(x)| \mathrm{d}x < +\infty$$

$L^2(\mathbb{R})$: **square-integrable functions**

$$f \in L^2(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x < +\infty$$

$L^\infty(\mathbb{R})$: **bounded functions**

$f \in L^\infty(\mathbb{R}) \Leftrightarrow \exists C \in \mathbb{R}: |f(x)| \leq C$ p.p.

Inclusions

$$L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

Proof $\int |f|^2 \leq \int \|f\|_\infty |f| = \|f\|_\infty \|f\|_1$

$f \in L^2(\mathbb{R})$ and $\exists A: \forall |x| > A, f(x) = 0 \Rightarrow f \in L^1(\mathbb{R})$

Proof

$$\int |f| = \int_{|f|>1} |f| + \int_{0<|f|\leq 1} |f| + \int_{|f|=0} |f|$$

$$\leq \int_{|f|>1} |f|^2 + \int_{0<|f|\leq 1} 1 \leq \|f\|_2^2 + 2A$$

Rules

$$L^k \cdot L^\infty \to L^k$$
$$L^2 \cdot L^2 \to L^1$$

$$L^k * L^1 \to L^k$$
$$L^2 * L^2 \to L^\infty$$

Continuos-time Fourier Transform for absolutely integrable functions

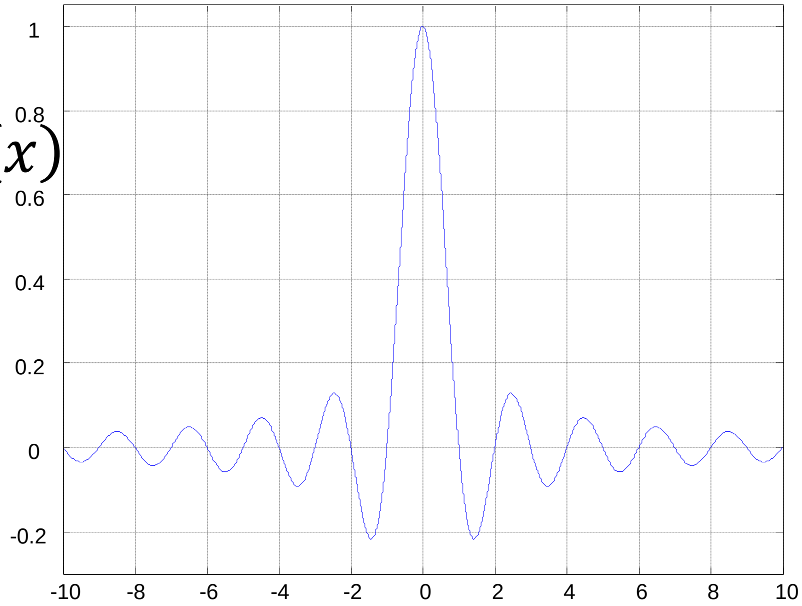
$f \in L^1(\mathbb{R})$, its CTFT is $\hat{f}$ or $\mathcal{F}(f)$:

$$\forall \nu \in \mathbb{R}, \hat{f}(\nu) = [\mathcal{F}(f)](\nu) = \int_{\mathbb{R}} f(x) e^{-2i\pi\nu x} dx$$

$\hat{f}$ is continue, bounded($\|\hat{f}\|_\infty \leq \|f\|_1$) and **the limit of *f* as *x* approaches infinity is *0***.

Example: compute the CTFT of $\mathbb{I}_{\left[-\frac{1}{2},\frac{1}{2}\right[}(x)$

$$\hat{f}(\nu) = \frac{\sin \pi\nu}{\pi\nu} = \mathrm{sinC}(\pi\nu)$$



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Inversion

If $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ the inverse transform is

$$\bar{\mathcal{F}}[f](\nu) = \int_{\mathbb{R}} f(t) e^{2i\pi t\nu} \mathrm{d}t$$

then $\bar{\mathcal{F}}[\mathcal{F}(f)] = f$

Point-by-point identity for, $L^1(\mathbb{R})$ norm-identity in $L^2(\mathbb{R})$

Example : compute the transform of $g(x) = [\mathrm{sinC}(\pi x)]^2$

$$\mathcal{F}[g] = \mathcal{F}[h \cdot h] = \mathcal{F}[h] * \mathcal{F}[h]$$

Properties

Land :

$f, g \in L^1 \cup L^2, \phi(x) = e^{2i\pi\nu_0 x}, f^y(x) = f(x - y)$

$\mathcal{F}(f * g) = \hat{f}\hat{g}$

$\mathcal{F}(fg) = \hat{f} * \hat{g}$

$\mathcal{F}(\phi f) = \hat{f}(\nu - \nu_0)$

$\mathcal{F}(f^y)(\nu) = \hat{f}(\nu)e^{-2i\pi\nu y}$

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Properties

Symmetry : If $\forall x \in \mathbb{R},\ f(x) = f(-x)$,
then $\forall \nu \in \mathbb{R},\ \hat{f}(\nu) = \hat{f}(-\nu)$

Hermitian Symmetry : If f is real, alors
$\forall \nu \in \mathbb{R}, \hat{f}(-\nu) = \overline{\hat{f}(\nu)}$

If f is real and even , $\hat{f}$ is real and even

Re-normalisation of time

$$\mathcal{F}[f(\lambda x)](\nu) = \frac{1}{\lambda}\hat{f}\left(\frac{x}{\lambda}\right)$$

Example : compute the CTFT of $\mathbb{I}_{\left[-\frac{A}{2},\frac{A}{2}\right]}$

$$\hat{f}(\nu) = A\ \mathrm{sinC}(A\pi\nu)$$

Properties

$$\forall k \in \{0,1,\dots K\}, x^k f(x) \in L^1(\mathbb{R}) \Rightarrow$$

$\hat{f} \in \mathcal{C}^{(k)}(\mathbb{R})$ and $\hat{f}^{(k)}(\nu) = \mathcal{F}[(-2i\pi x)^k f(x)]$

$\forall k \in \{0,1,\dots K\}, f \in \mathcal{C}^{(k)}(\mathbb{R})$,

$f^{(k)}(x) \in L^1(\mathbb{R})$

$\Rightarrow$

$\forall k \in \{0,1,\dots K\}, \lim_{|\nu|\to\infty} \nu^k \hat{f}(\nu) = 0$

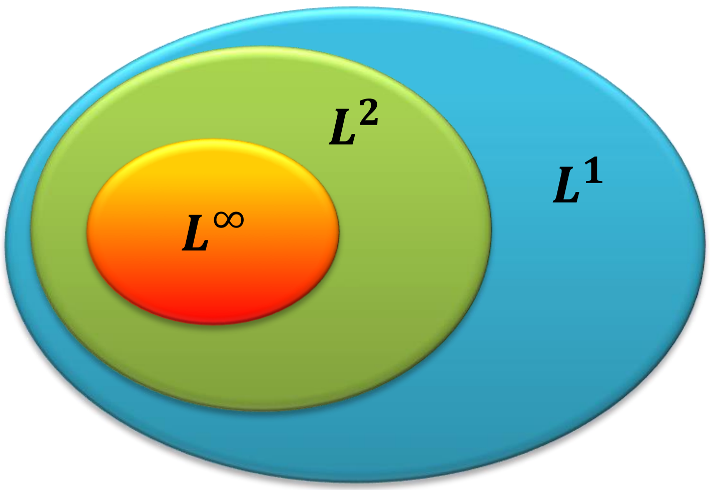
and $\mathcal{F}\left[f^{(k)}\right](\nu) = (2i\pi x)^k \hat{f}(\nu)$

Fourier Series

If $f \in L^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right[\right)$, the FS is the sequence c_k:

$$\forall k \in \mathbb{Z},\ c_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-2i\pi k t} \mathrm{d}t$$

c_k tends to zero for $k \to \pm\infty$



Properties

$f, g \in L^p\left(\left[-\frac{1}{2}, \frac{1}{2}\right[\right)$, c, d are their FS.

$$\mathcal{F}\left(e^{2i\pi mt}\right) = \delta_{k-m}$$

$$p = 1, \qquad \mathcal{F}(f * g) = cd$$

$$p = 2, \qquad \mathcal{F}(fg) = c * d$$

$$\mathcal{F}\left[f(t)e^{2i\pi k_0 t}\right](k) = c(k - k_0)$$

$$\mathcal{F}[f(t - y)](k) = c(k)e^{-2i\pi ky}$$

Properties

Hermitian Symmetry : If f is real, $c_{-k} = \bar{c_k}$

Symmetry : If $\forall x \in \mathbb{R}, f(x) = f(-x),\ c_{-k} = c_k$

If f is real and even , its FS is also real and even

Parseval : If $f \in L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right)$ and c_k is its FS, then $c_k \in l^2(\mathbb{Z})$ et

$$\|f\|_2 = \|c\|_2$$

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Inversion

$f \in L^1\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right)$ and $c \in l^1$

$$\Rightarrow \forall t \in \left[-\frac{1}{2},\frac{1}{2}\right[, \sum_{k\in\mathbb{Z}} c_k e^{2i\pi tk} = f(t)$$

$f(-\cdot)$ is the DTFT of c_k

$f \in L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right) \Rightarrow \forall t \in \left[-\frac{1}{2},\frac{1}{2}\right[, \sum_{k\in\mathbb{Z}} c_k e^{2i\pi tk}$ tends to $f(t)$ in norm:

$$\lim_{K\to\infty} \int_{\mathbb{R}} \left| f(t) - \sum_{k=-K}^{K} c_k e^{2i\pi \mathrm{tk}} \right| dt = 0$$

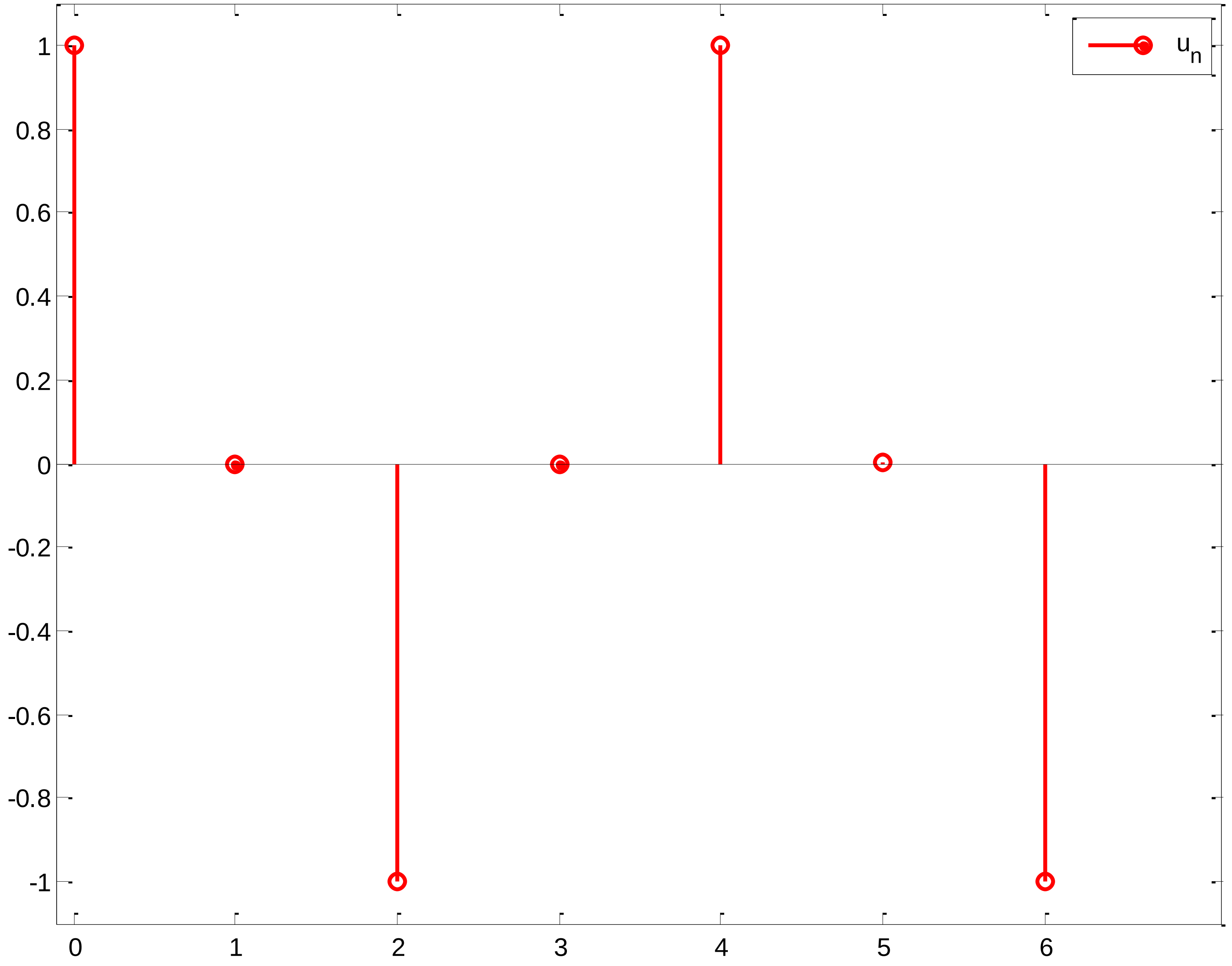
Properties

If

$$f \in L^1\left(\left[-\frac{1}{2},\frac{1}{2}\right[\right) \cap \mathcal{C}^{(N)},$$

then :

$$\lim_{|k|\to\infty} k^N c_k = 0$$



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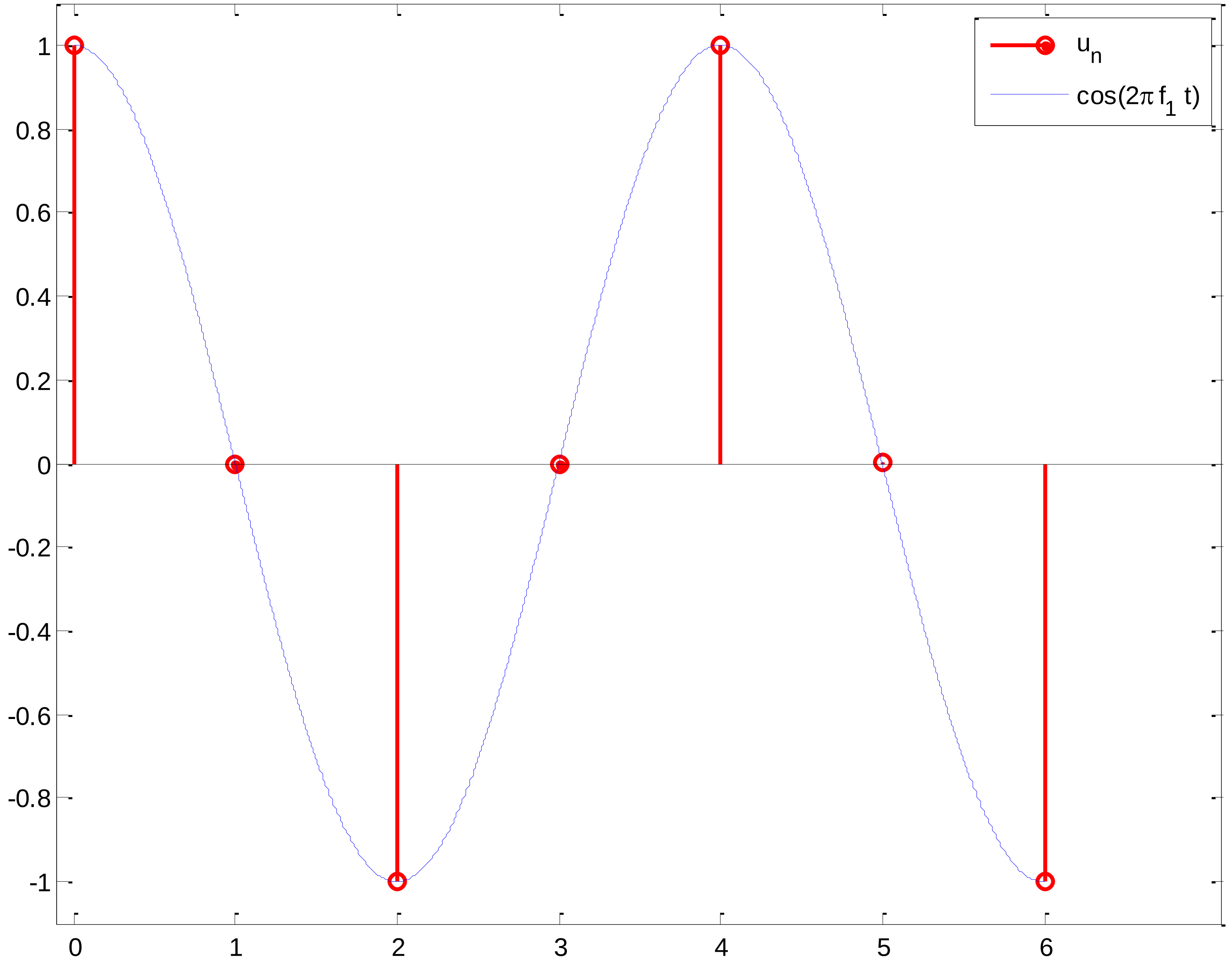

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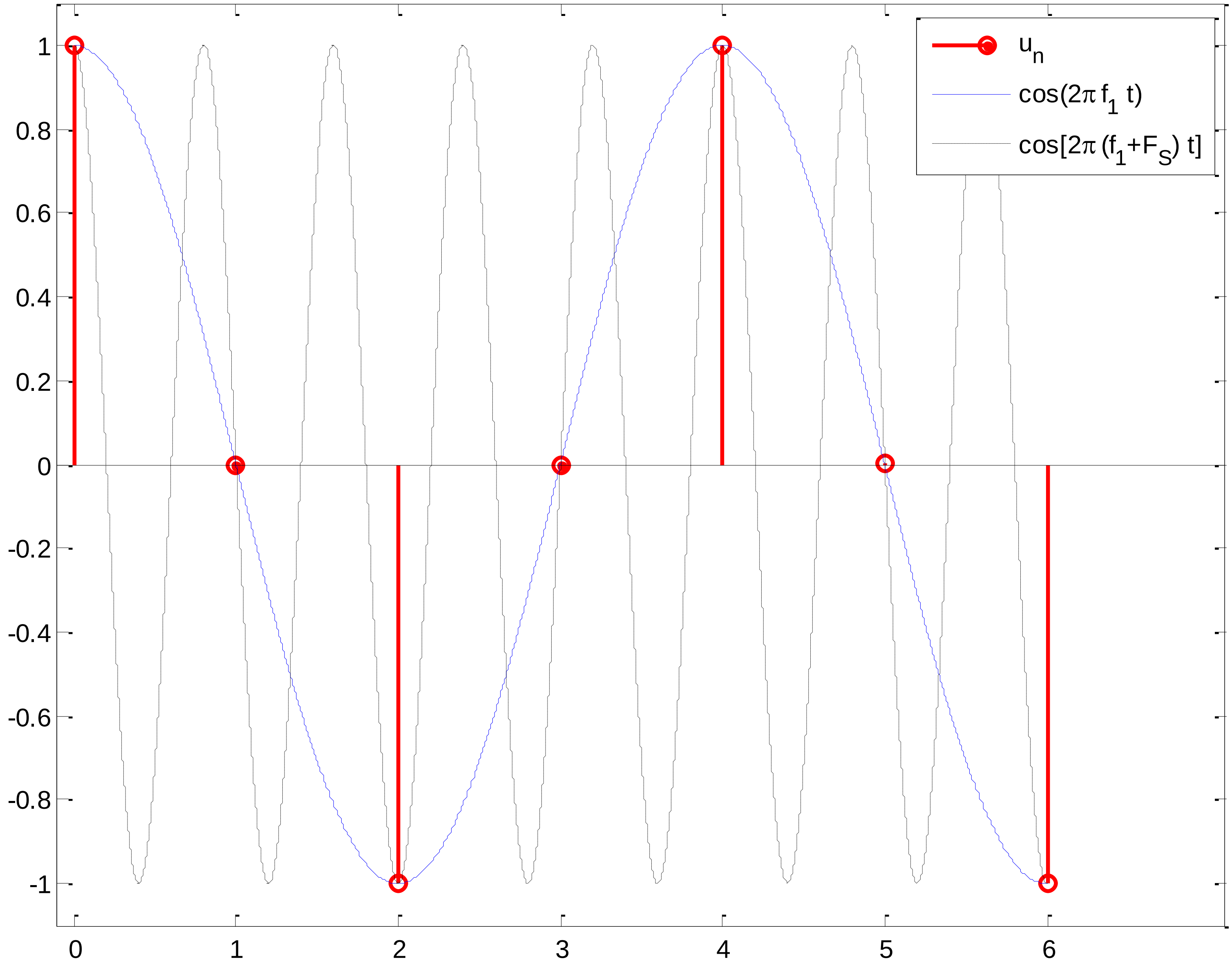
u_n
$\cos(2\pi f_1 t)$



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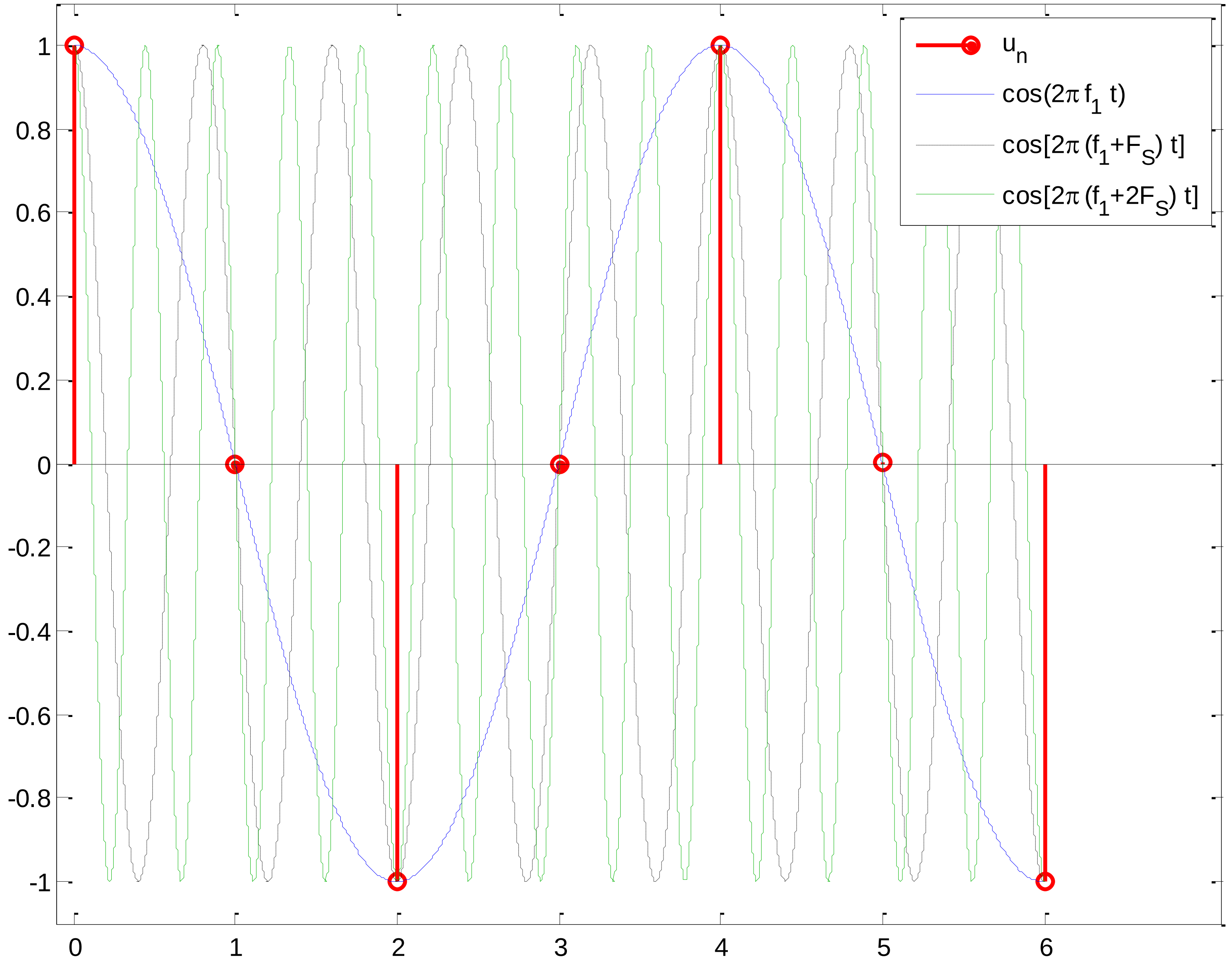
$\cos(2\pi f_1 t)$
$\cos[2\pi (f_1+F_S) t]$



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u_n
$\cos(2\pi f_1 t)$
$\cos[2\pi (f_1+F_S) t]$
$\cos[2\pi (f_1+2F_S) t]$

Poisson formula

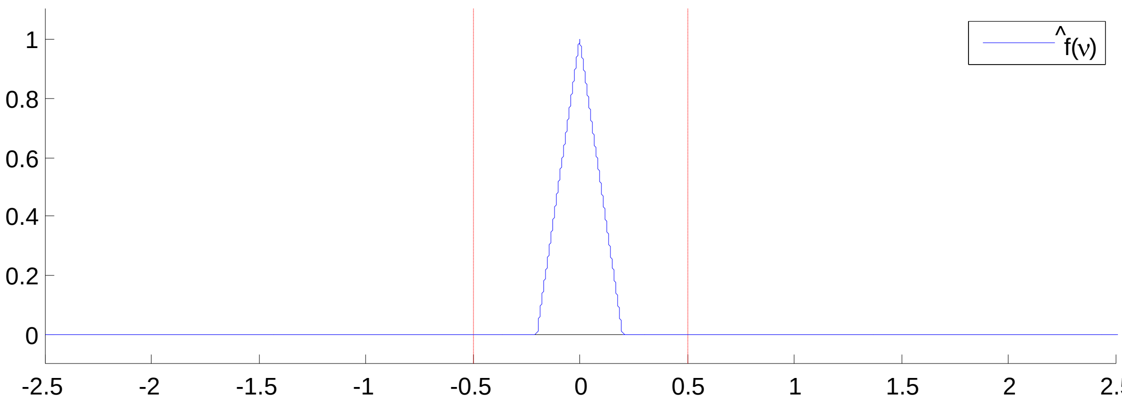
$$f \in L^1(\mathbb{R}), \hat{f} \in L^1(\mathbb{R}), f(n) \in l^1, \hat{f}(m) \in l^1 \Rightarrow$$

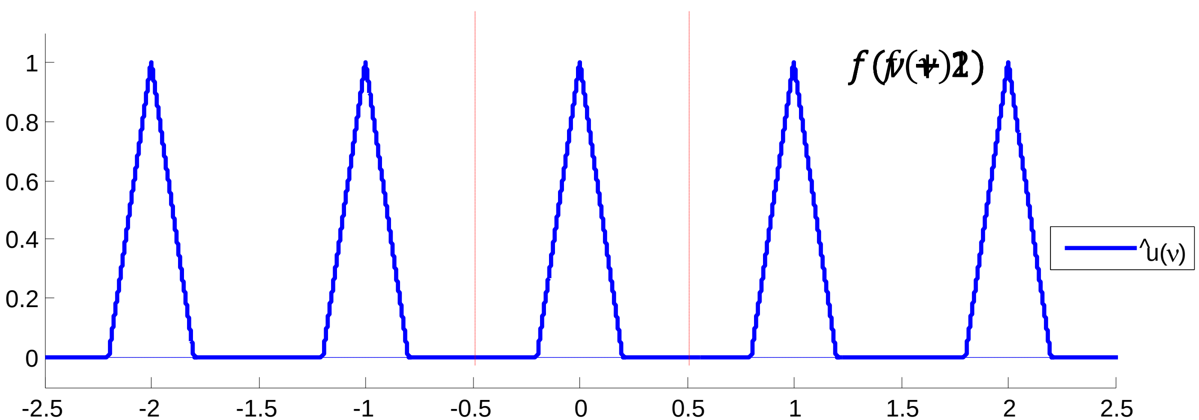
$$\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[, \qquad \sum_{m \in \mathbb{Z}} f(m) e^{-2i\pi m\nu} = \sum_{n \in \mathbb{Z}} \hat{f}(\nu + n)$$

Poisson formula

- $\sum_m f(m) = \sum_n \hat{f}(n)$
- $\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[\quad \hat{u}(\nu) = \sum_{n\in\mathbb{Z}} \hat{f}(\nu + n)$
- Aliasing

$$\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[\quad \hat{u}(\nu) = \sum_{n \in \mathbb{Z}} \hat{f}(\nu + n)$$





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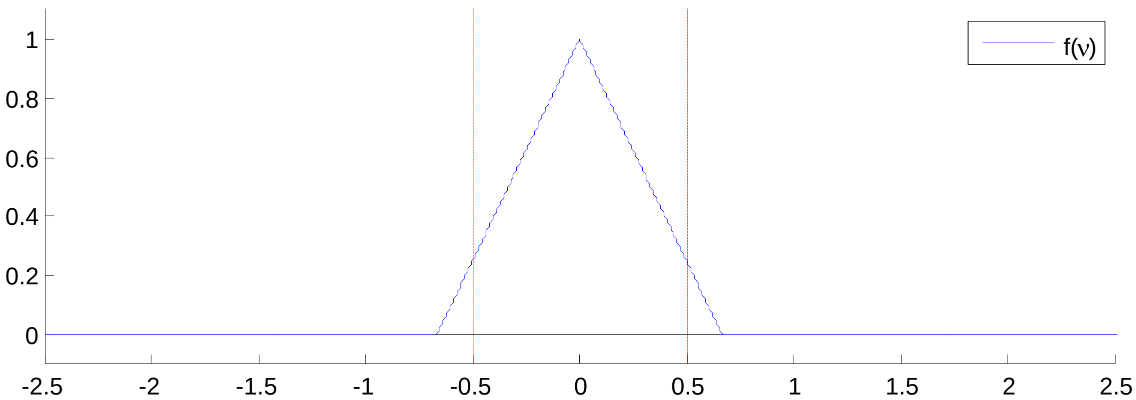
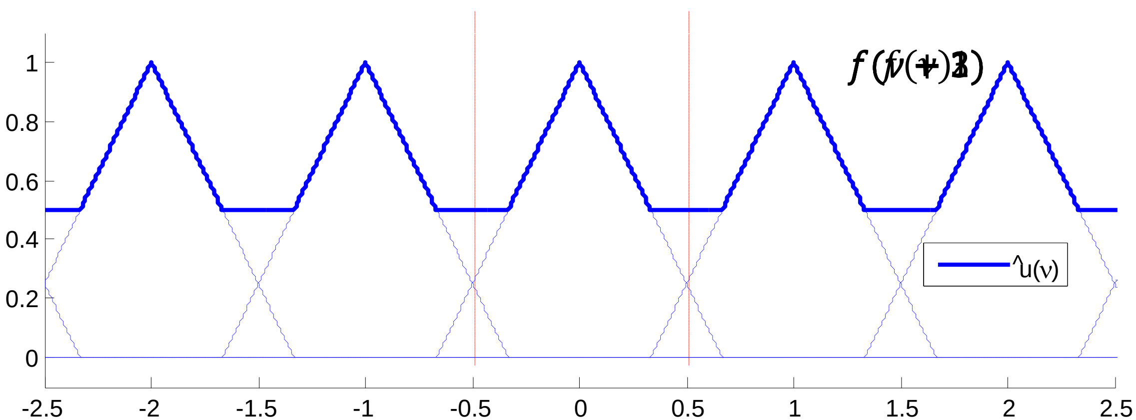
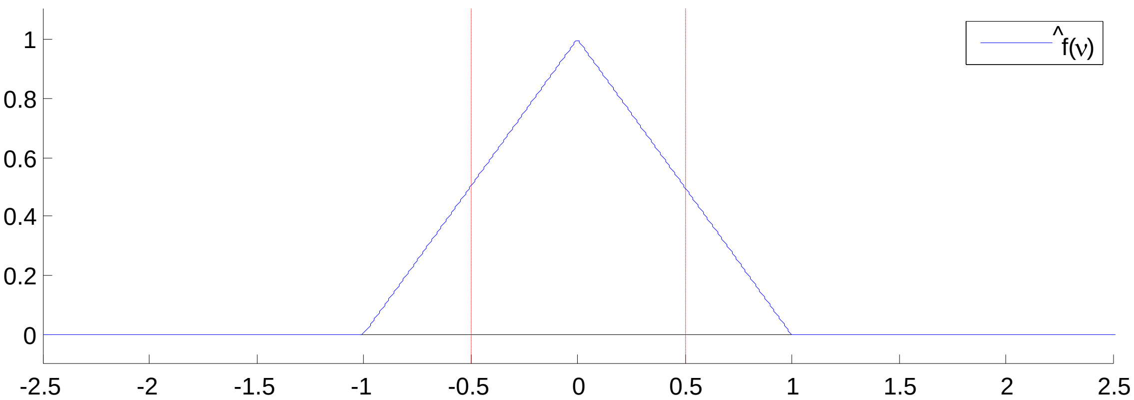
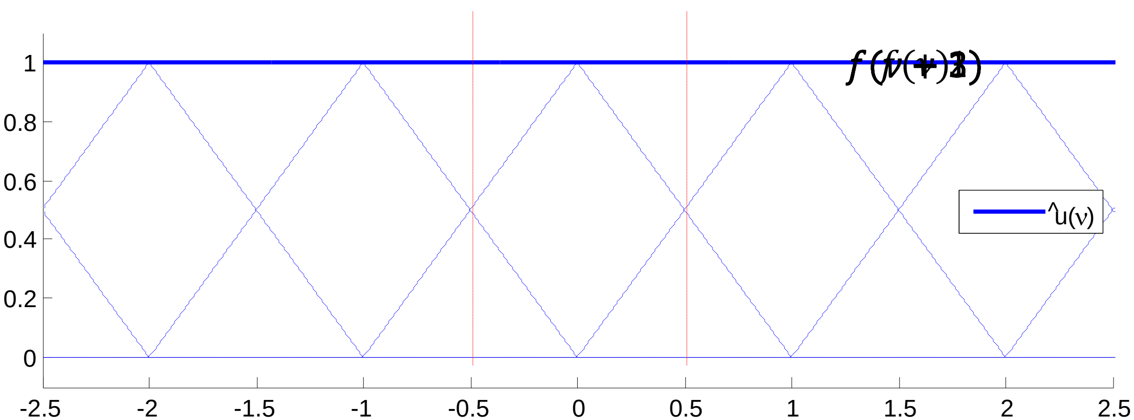
$$\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[\quad \hat{u}(\nu) = \sum_{n \in \mathbb{Z}} \hat{f}(\nu + n)$$





$$\forall \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right[\quad \hat{u}(\nu) = \sum_{n \in \mathbb{Z}} \hat{f}(\nu + n)$$





Shannon sampling theorem

$f \in L^1(\mathbb{R}),\ \forall \nu \notin \left[-\frac{1}{2},\frac{1}{2}\right]\ \hat{f}(\nu) = 0$ and $f(n) \in l^1$

then

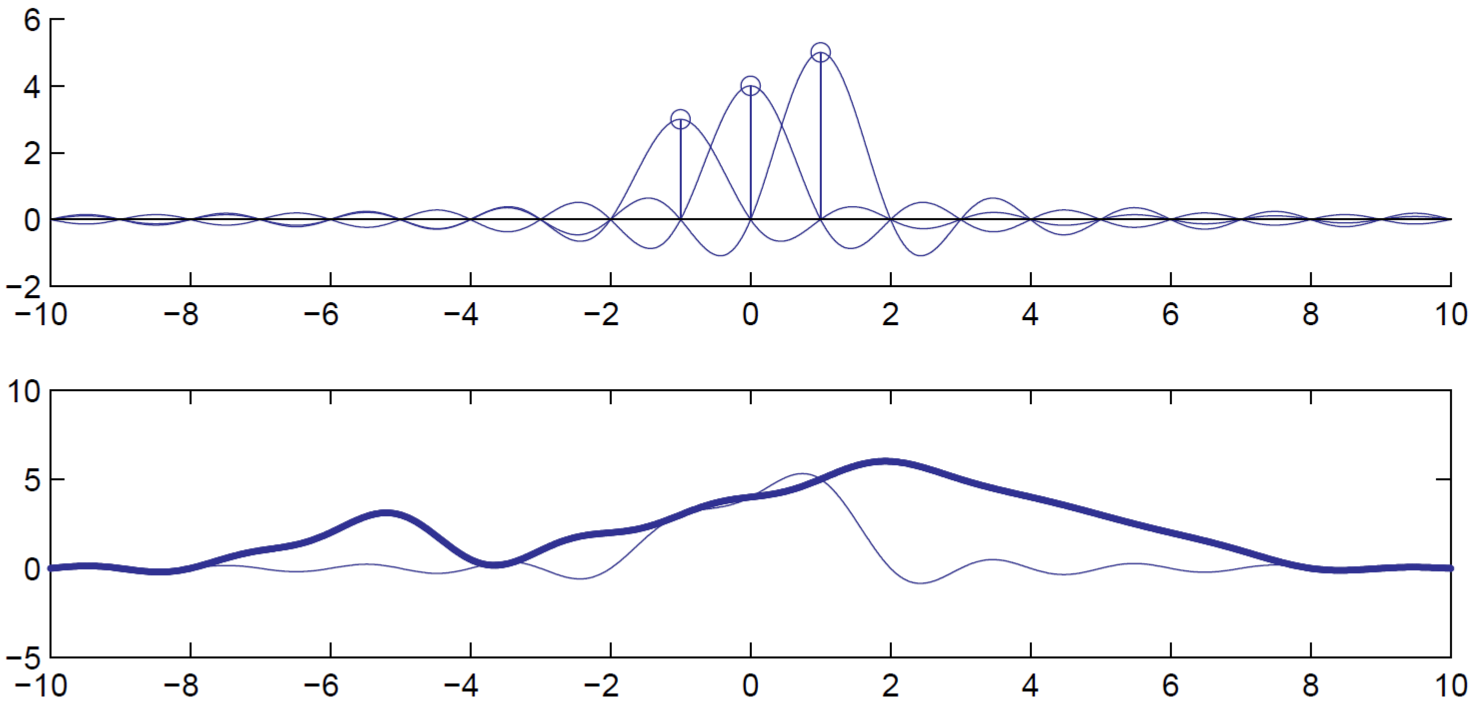
$$\forall t \in \mathbb{R}, f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinC}[\pi(t-n)]$$

And

$$\forall \nu \in \left[-\frac{1}{2},\frac{1}{2}\right[, \sum_{m\in\mathbb{Z}} f(m)e^{-2i\pi\nu m} = \hat{f}(\nu)$$

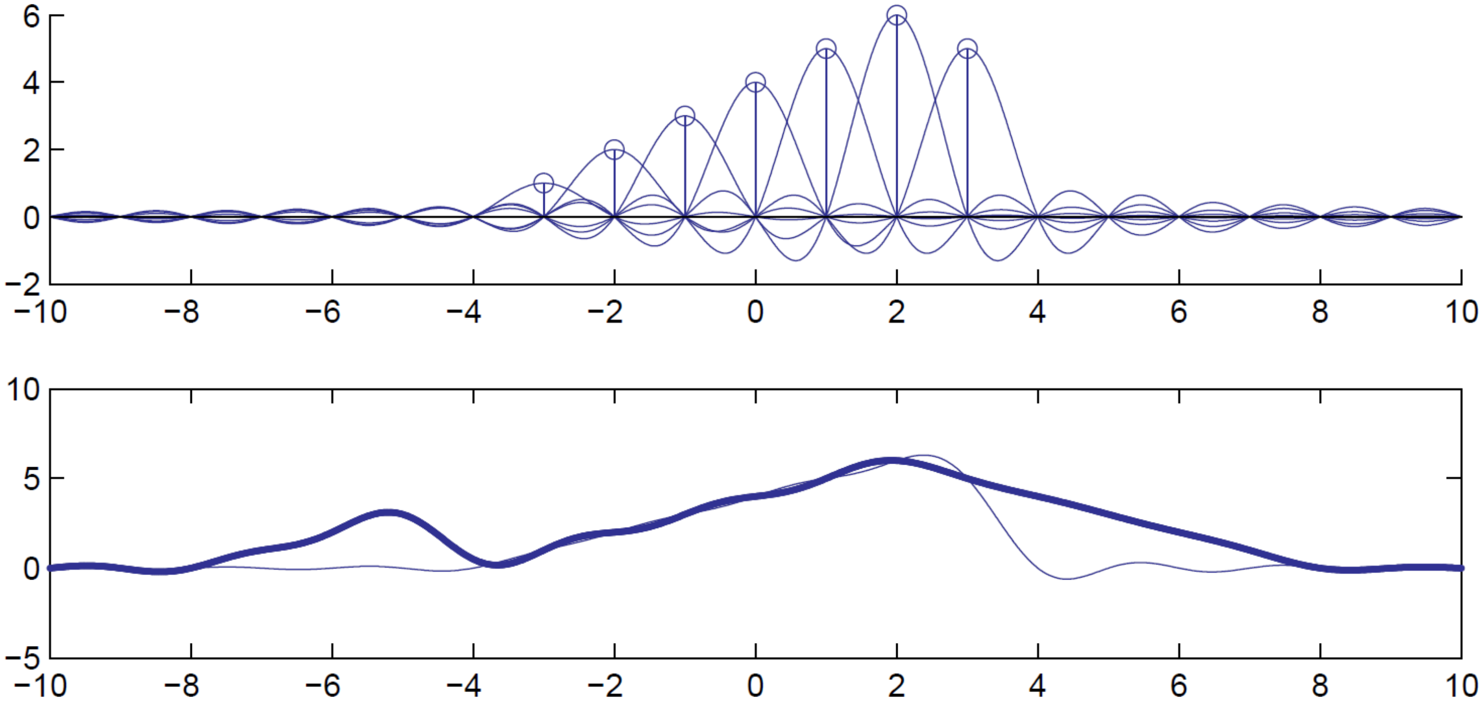
Perfect reconstruction

$$\forall t \in \mathbb{R}, f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinC}[\pi(t-n)]$$



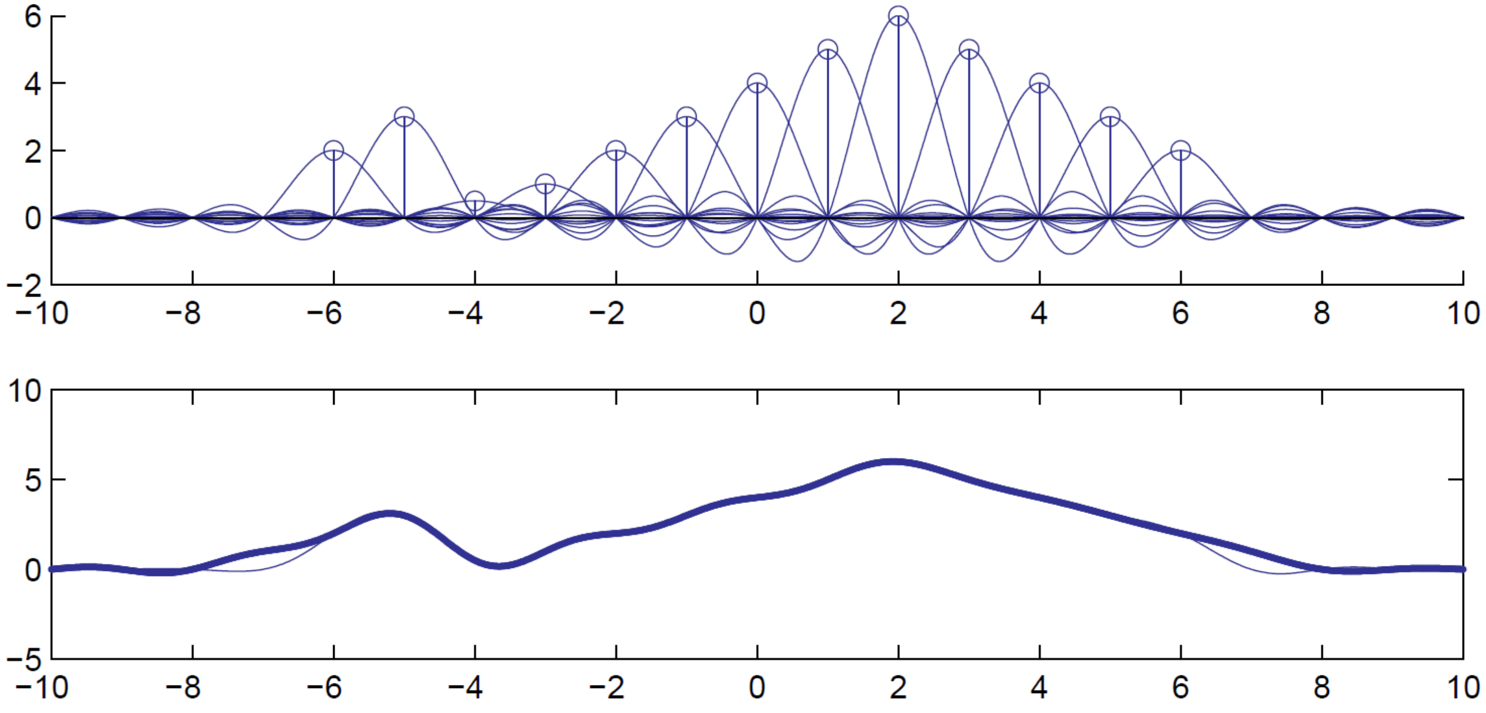
Perfect reconstruction

$$\forall t \in \mathbb{R}, f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinC}[\pi(t-n)]$$



Perfect reconstruction

$$\forall t \in \mathbb{R}, f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinC}[\pi(t-n)]$$



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Other reconstructions

$$\forall t \in \mathbb{R}, f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinC}[\pi(t-n)]$$

Problems

Other reconstructions

$$\forall t \in \mathbb{R}, g(t) = \sum_{n \in \mathbb{Z}} f(n) h(t-n)$$

Trunkated sinC : $h(t) = \mathrm{sinC}(t)\mathbb{I}_{(-T,T)}(t)$

Error is $\mathrm{O}\left(\frac{1}{T}\right)$

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Other reconstructions

Polynomial interpolation polynomiale : $g(t)$ is a polynomial and $\forall n \in \mathbb{Z}, g(n) = f(n)$

Zero-order Interpolation : $h(t) = h_0(t) = \mathbb{I}_{\left[-\frac{1}{2},\frac{1}{2}\right]}(t)$

$$\forall t \in \mathbb{R}, g(t) = \sum_{n \in \mathbb{Z}} f(n) h_0(t-n) \Leftrightarrow$$
$$\forall t \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right], g(t) = f(m)$$

Other reconstructions

First order Interpolation :

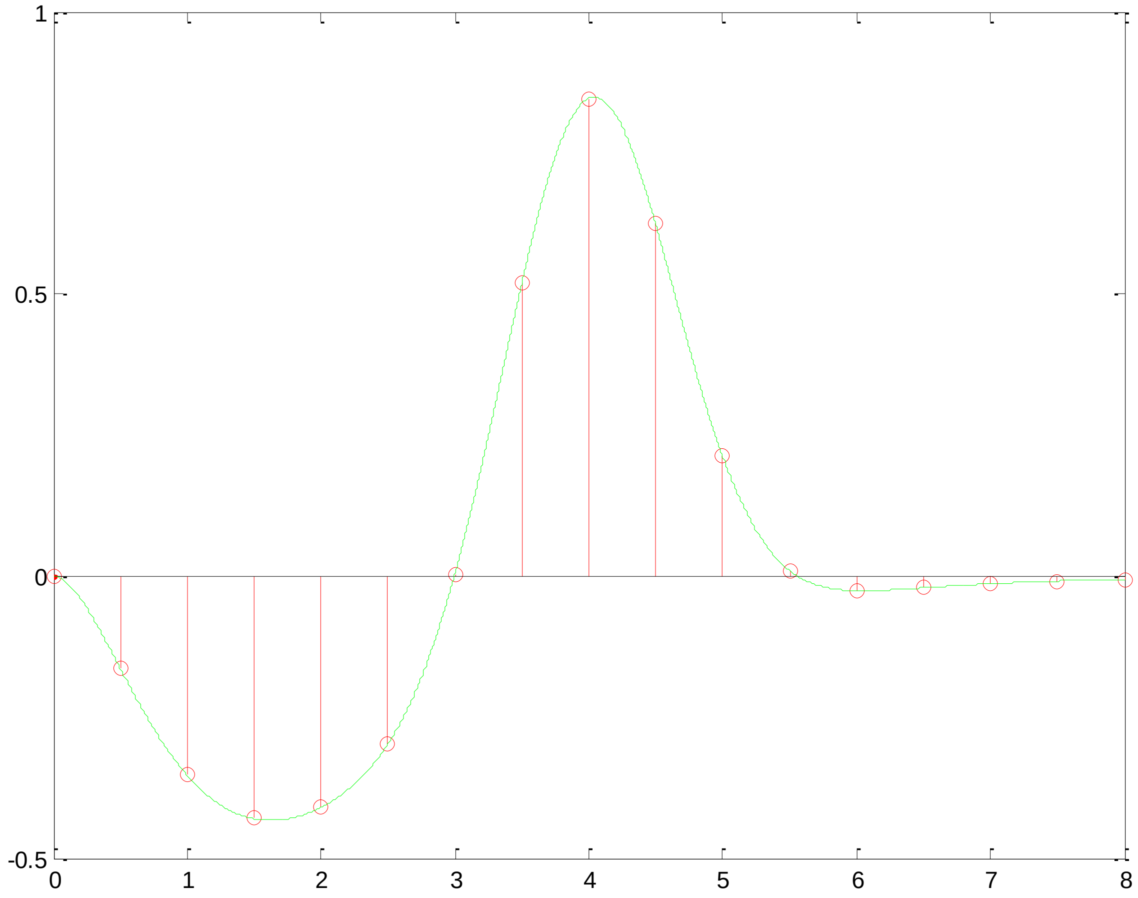
$$h(t) = h_1(t) = (h_0 * h_0)(t)$$
$$= (1 - |t|)\mathbb{I}_{[-1,1]}(t)$$

$$\forall t \in \mathbb{R}, g(t) = \sum_{n \in \mathbb{Z}} f(n) h_1(t - n)$$

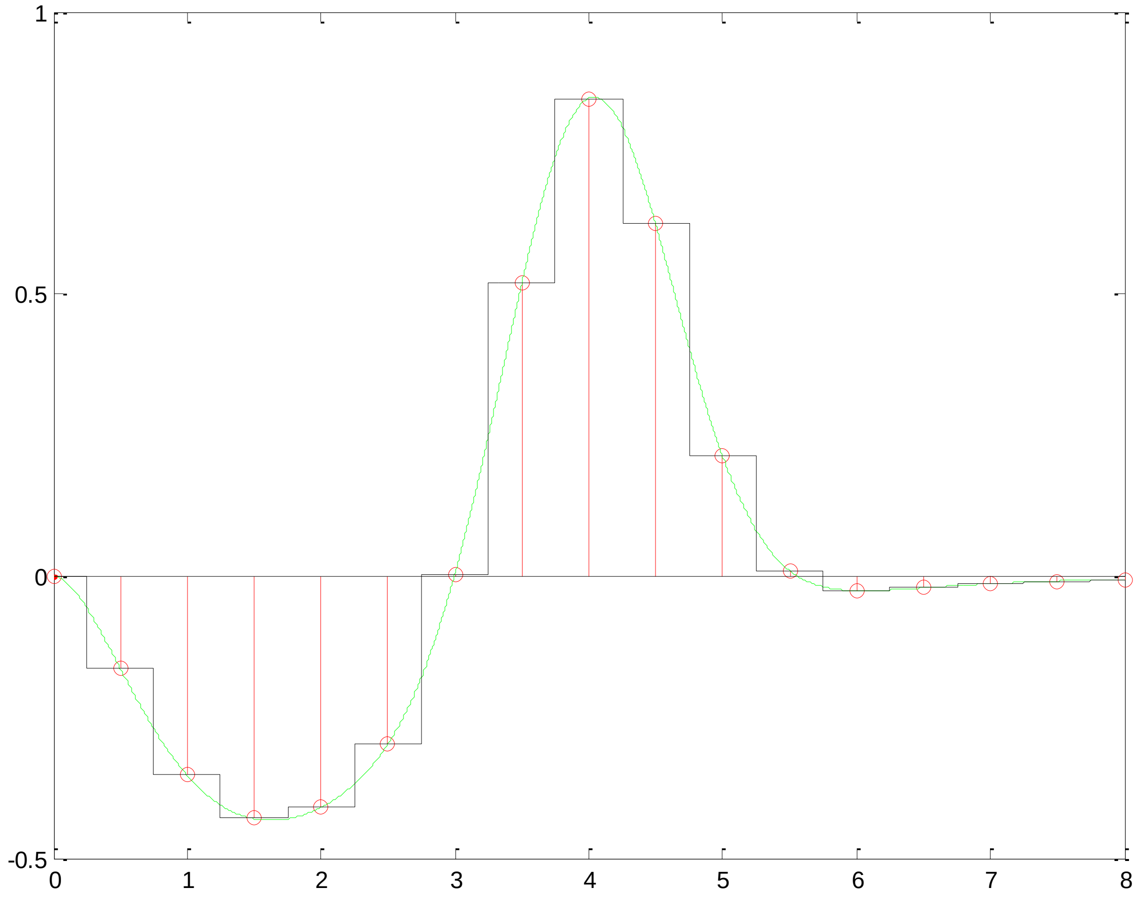
Let $\theta = t - m.$

$\forall t \in [m, m+1], \theta \in [0,1]$

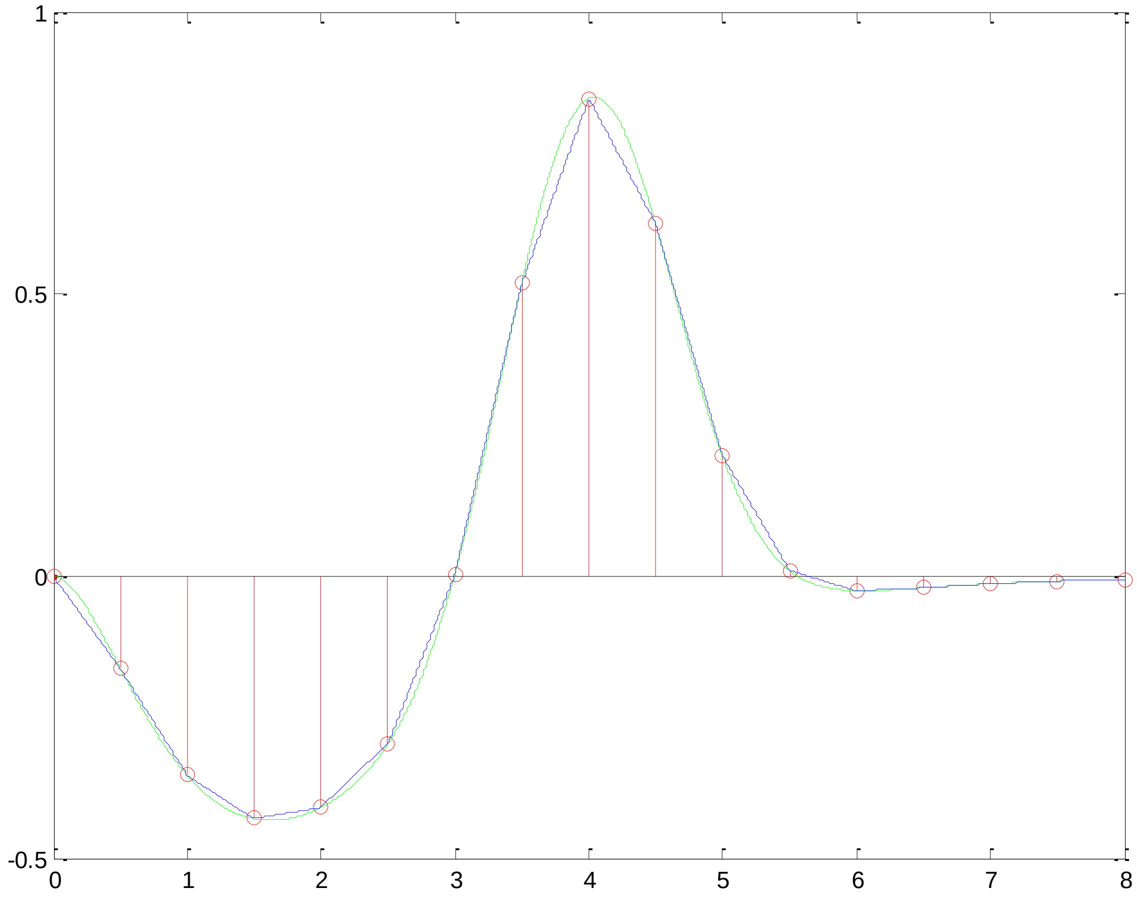
Other reconstructions



Other reconstructions



Other reconstructions



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Reconstruction error

f satisfies Poisson

$$g(t) = \sum_{n} f(n)h(t-n)$$

$$\hat{g}(\nu) = \sum_{n\in\mathbb{Z}} f(n)\hat{h}(\nu)e^{-2i\pi\nu n} = \hat{h}(\nu)\sum_{m\in\mathbb{Z}} \hat{f}(\nu+m)$$

- Linear distortion $\hat{h}(\nu)$
- Aliasing

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Reconstruction error

f satisfies Poisson and its spectrum has support in $\left[-\frac{1}{2},\frac{1}{2}\right]$

$g(t)=\sum_n f(n)h(t-n)$ and $\hat{g}(\nu)=\sum_{n\in\mathbb{Z}} f(n)\hat{h}(\nu)e^{-2i\pi\nu n}=\hat{h}(\nu)\sum_{m\in\mathbb{Z}}\hat{f}(\nu+m)$

$$\|f-g\|_2^2=\left\|\hat{f}-\hat{g}\right\|_2^2=$$

$$=\int_{-1/2}^{1/2}\left|\hat{f}(\nu)-\hat{g}(\nu)\right|^2 d\nu+\sum_{n\neq 0}\int_{n-1/2}^{n+1/2}\left|\hat{f}(\nu)-\hat{g}(\nu)\right|^2 d\nu$$

$$=\int_{-1/2}^{1/2}\left|\hat{f}(\nu)\right|^2\left|1-\hat{h}(\nu)\right|^2 d\nu+\sum_{n\neq 0}\int_{n-1/2}^{n+1/2}|\hat{g}(\nu)|^2 d\nu$$

$$=\int_{-1/2}^{1/2}\left|\hat{f}(\nu)\right|^2\left|1-\hat{h}(\nu)\right|^2 d\nu+\sum_{n\neq 0}\int_{-1/2}^{+1/2}|\hat{g}(\nu-n)|^2 d\nu$$

$$=\int_{-1/2}^{1/2}\left|\hat{f}(\nu)\right|^2\left|1-\hat{h}(\nu)\right|^2 d\nu+\sum_{n\neq 0}\int_{-1/2}^{+1/2}\left|\hat{h}(\nu-n)\hat{f}(\nu)\right|^2 d\nu$$

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Reconstruction error

$$\|f-g\|_2^2 =$$
$$= \int_{-1/2}^{1/2} \left|\hat{f}(\nu)\right|^2 \left|1-\hat{h}(\nu)\right|^2 d\nu + \int_{-1/2}^{+1/2} \left|\hat{f}(\nu)\right|^2 \sum_{n\neq 0} \left|\hat{h}(\nu-n)\right|^2 d\nu$$

Best reconstruction

Let $f \in L^2$ **non band-limited**

Let $g(t) = \sum_n u_n \operatorname{sinC}[\pi(t-n)]$ and ideal DAC

What u_n minimizes $\|f - g\|_2^2$?

Best reconstruction

Let $f_B = f * \mathrm{sinC}^{\pi} \in L^2$: is band-liminted

Let $f_H = f - f_B$ that is, $f = f_H + f_B$

$\forall g$ exprimable as $\sum \mathrm{sinC}$ (band-minited), we have

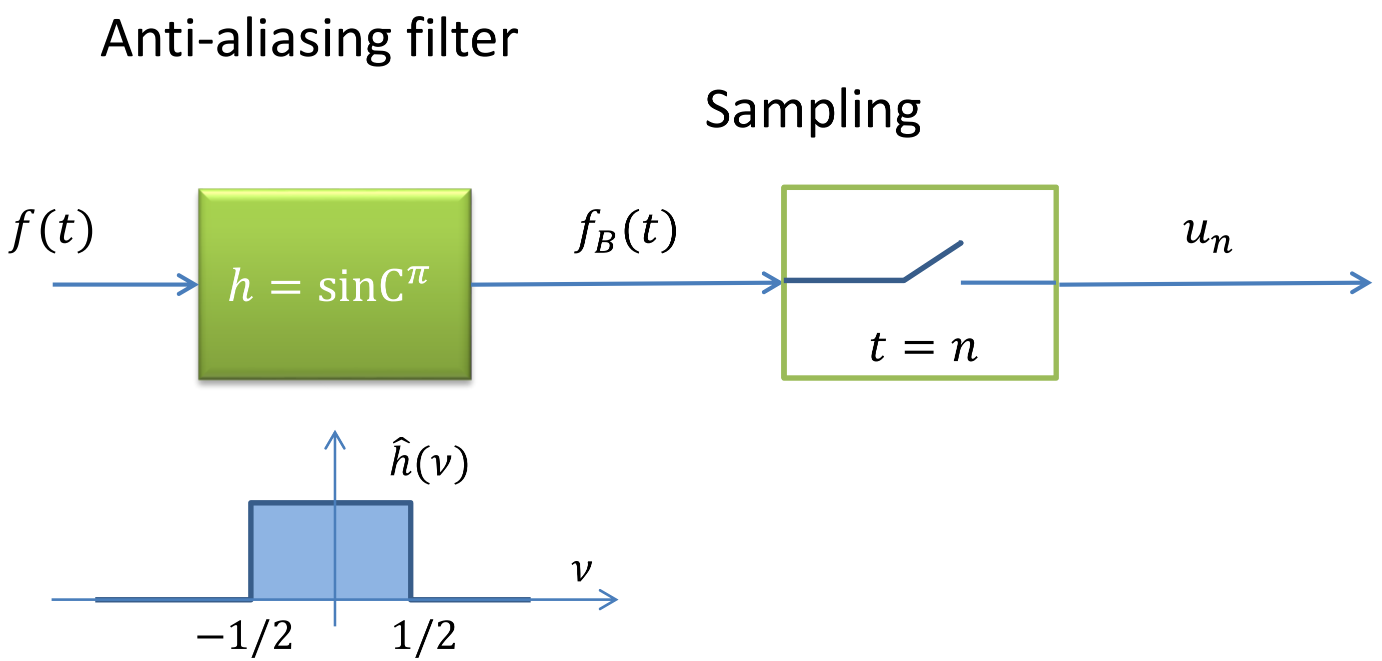
$$\|f - g\|_2^2 = \left\|\hat{f} - \hat{g}\right\|_2^2 = \left\|\hat{f}_H + \hat{f}_B - \hat{g}\right\|_2^2$$

$$= \left\|\hat{f}_H\right\|_2^2 + \left\|\hat{f}_B - \hat{g}\right\|_2^2 + 2\langle \hat{f}_H, \hat{f}_B - \hat{g}\rangle =$$

$$= \left\|\hat{f}_H\right\|_2^2 + \left\|\hat{f}_B - \hat{g}\right\|_2^2 = \|f_H\|_2^2 + \|f_B - g\|_2^2$$

Since f_B and g satisfy Shannon, $g(n) = f_B(n)$ is the best choice and $u_n = f_B(n)$

Sampling system



Time normalization

If f is sampled with period T_e, we introduce $g(x) = f(T_e x)$

g is sampled with period 1, so

1. $\forall |\nu| > \frac{1}{2}, \hat{g}(\nu) = 0$
2. $g(x) = \sum_n g(n) \mathrm{sinC}[\pi(x-n)]$
3. $\sum_m g(m) e^{-2i\pi\nu m} = \sum_n \hat{g}(\nu + n)$

Moreover

$$g(x) = f(T_e x) \qquad f(x) = g\left(\frac{x}{T_e}\right)$$

$$\hat{g}(\nu) = \frac{1}{T_e}\hat{f}\left(\frac{\nu}{T_e}\right) \qquad \hat{f}(\nu) = T_e \hat{g}\left(\frac{\nu}{F_e}\right)$$

Normalisation

1. $\hat{f}(\nu) = T_e \hat{g}\left(\frac{\nu}{F_e}\right) = 0 \ \forall \frac{|\nu|}{F_e} > \frac{1}{2} \Leftrightarrow |\nu| > \frac{F_e}{2}$
2. $f(t) = g(t/T_e) =$
   $\sum_n g(n) \mathrm{sinC}\left[\pi\left(\frac{t}{T_e} - n\right)\right] f(t) =$
   $\sum_n f(nT_e) \mathrm{sinC}\left[\frac{\pi}{T_e}(t - nT_e)\right]$
3. $\sum_m f(mT_e) e^{-2i\pi\nu m} = \frac{1}{T_e} \sum_n \hat{f}\left(\frac{\nu + n}{T_e}\right) =$
   $\frac{1}{T_e} \sum_n \hat{f}[F_e(\nu + n)]$

Sampling system

Anti-aliasing filter
$F_E/2$

Sampling at T_E